

Role of the impurity-potential range in disordered d -wave superconductors

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Abstract

We analyze how the range of disorder affects the localization properties of quasiparticles in a two-dimensional d -wave superconductor within the standard non-linear σ -model approach to disordered systems. We show that for purely long-range disorder, which only induces intra-node scattering processes, the approach is free from the ambiguities which often beset the disordered Dirac-fermion theories, and gives rise to a Wess-Zumino-Novikov-Witten action leading to vanishing density of states and finite conductivities. We also study the crossover induced by internode scattering due to a short range component of the disorder, thus providing a coherent non-linear σ -model description in agreement with all the various findings of different approaches.

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I. INTRODUCTION

The low-temperature quasiparticle transport in two-dimensional d -wave superconductors like cuprates is a fascinating issue due to the presence of four nodes in the energy spectrum of the Bogoliubov quasiparticles, around which the low-lying excitations have a Dirac-like dispersion. Within the self-consistent Coherent-Phase-Approximation in the limit of weak disorder, spin and thermal conductivities are found to be related by a Wiedemann-Franz law and to acquire universal values which do not depend on the disorder strength.¹ Inclusion of quantum interference in the framework of the standard non-linear σ -model approach to disordered systems^{2,3} leads to a variety of universality classes^{4,5,6,7,8,9,10,11}. In the “generic” case of short-ranged non magnetic impurities full localization of Bogoliubov quasiparticles is predicted.⁵

Nevertheless, with the only exceptions of YBCO(124)¹² and $\text{Pr}_{2-x}\text{Ce}_x\text{CuO}_4$ ¹³, experiments in cuprates materials like YBCO(123)^{14,15,16}, BSCCO(2212)¹⁵ and LSCO¹⁷ do not show any evidence of strong or even weak localization in the superconducting phase down to 0.1 Kelvin. Various physical effects may be invoked to explain the disagreement between theory and experiments.

For instance one may argue that the origin of the discrepancy are spin-flip scattering events, even though the systems are nominally free from magnetic impurities. Indeed, in the presence of spin-flips, the non-linear σ -model predicts that quantum interference has a delocalizing effect⁹. Alternatively, or in addition, strong dephasing processes might set the temperature scale for the onset of localization effects below the experimentally accessed region. This question has not been settled yet¹⁸. Residual interactions among quasiparticles can also favor delocalization^{9,11}, even though, in the weak disorder limit, they are expected to be less effective since their coupling is proportional to the density of states which, already in the Born approximation, is very small.

Another possible explanation invokes the range of the impurity potential. In the case of purely long range disorder, forward processes dominate and scattering occurs mainly within each node. In the extreme case of intra-node scattering only, it has been shown^{19,20} that the density of states behaves quite differently from the isotropic-scattering case. In addition the eigenstates have been argued²⁰ to be delocalized, unlike for short-range impurity potential. Even though real disorder will always have an isotropic component which provides scattering

among all four nodes, and eventually drive the system to localization, one might argue that a large value of the intranode with respect to the internode scattering could lower the crossover temperature for the appearance of localization precursor effects. A sizeable amount of long range disorder has been indeed argued to be present in cuprates on the basis of STM and microwave conductivity experiments²¹. This is not surprising since superconducting cuprates are intrinsically disordered by the out-of-plane charge dopants which mainly provide a long-range disordered potential. Further doping with iso-valent impurities which substitute in-plane Cu-ions only adds a short-range component on top of the pre-existing long-range tail of the disordered potential.

The results in the presence of purely intra-node impurity scattering have been obtained^{19,20} within an approach which is conceptually different from the standard non-linear σ -model approach to disordered systems. The latter starts from the Born approximation, i.e. from impurity-damped quasiparticles, and treats perturbatively what is beyond that, i.e. quantum interference effects on the diffusive motion. On the contrary, the alternative methods used in Refs. [19,20] do not rely on the Born approximation but just map the action of ballistic nodal-quasiparticles in the presence of disorder onto the action of one-dimensional (1d) fermions in the presence of an interaction, which is generated by the disorder average. Within the mapping, one of the two spatial dimensions transforms into the time coordinate of the 1d model while the other into the 1d spatial coordinate. The final model is then analyzed by abelian and non-abelian bosonization. The outcome of this analysis is that for purely intra-node or inter-opposite-node disorder, where essentially exact results can be obtained^{19,20}, the density of states is power-law vanishing at the chemical potential with an exponent which is disorder-dependent in the former case and universal in the latter. In both cases the results suggest that a diffusive quasiparticle motion never sets in, namely quasiparticles move ballistically down to zero energy. When the disorder also couples adjacent nodes, strong coupling arguments are invoked²⁰ which predict localization of quasiparticles and linearly vanishing density of states. Yet, even this case seems to suggest a scenario in which quasiparticle motion from ballistic turns directly into localized without crossing any intermediate diffusive regime.

The standard non-linear σ -model approach applies the replica trick to average over disorder from the outset. The resulting fermion interaction is then decoupled by introducing Q -matrix fields in terms of which an effective action is derived after integrating out the

fermions. The saddle point solution is just the Born approximation, which provides, in the case of nodal Bogoliubov quasiparticles, a finite density of states hence a finite damping. Finally, long-wavelength transverse fluctuations are taken with respect to the saddle point leading to a non-linear σ -model action for the Q -matrix fields. However, unlike in conventional disordered systems, in this particular case an additional term may appear in the non-linear σ -model action, namely a so-called Weyl-Zumino-Novikov-Witten (WZW) term. It was actually argued^{7,20,22} that two opposite nodes share the same WZW term, while two adjacent ones have opposite WZW terms. As a consequence, when the two pairs of opposite nodes are uncoupled by disorder the WZW term is effective and the β -function of the spin-conductance flows towards an intermediate-coupling fixed point, signaling a delocalized behavior. On the contrary, when disorder couples all nodes together, the two WZW terms cancel exactly and the non-linear σ -model has no more protection against flowing towards a zero-conductance strong coupling regime characterized by a linearly vanishing density of states⁶.

From the above discussion one might be lead to conclude that the agreement between the conclusions drawn with either methods is merely an accident which does not justify *per se* the conventional non-linear σ -model approach when dealing with Dirac fermions. The main objection against the conventional non-linear σ -model is that it is not appropriate to start from a symmetry breaking saddle-point solution, associated to a mean-field-like finite density of states, when the outcome of including fluctuations is a vanishing density of states. Put in a different language, it is hard to believe in a method which starts by assuming a diffusive behavior if at the end it is discovered that a diffusive regime never appears. This criticism could invalidate also the results obtained with isotropic scattering, even though in this case the dimensionless coupling of the non-linear σ -model can be made small²³ by assuming a large anisotropy of the Dirac dispersion (i.e. the velocities along and orthogonal to the Fermi surface). A related objection that can be raised is that the non-linear σ -model approach to disordered systems is commonly believed to be valid for length scales longer than the mean free path and energy scales smaller than the inverse relaxation time $1/\tau$. On the contrary, the solution of the intra-node scattering problem demonstrates that disorder starts to affect for instance the density of states at energies of the order of the superconducting gap, hence much bigger than $1/\tau$, namely in the regime when quasiparticle motion should

be still ballistic.

It is the scope of the present work to clear up these inconsistencies of the non-linear σ -model approach to disordered d -wave superconductors. We will demonstrate that the above, apparently contradicting, approaches can be actually reconciled. This is of particular interest since it provides further support to the standard non-linear σ -model approach based on the replica trick within the fermionic path-integral formalism, which remains so far one of the few available tools to deal with disorder in generic situations with a Fermi sea of interacting quasiparticles. Let us briefly summarize our main results.

First we are going to present a simple and straightforward way to extract the WZW term. Indeed it is well known how to derive the WZW term within field theories defined on a continuous space with Dirac-like dispersing particles. However in disordered lattice models the existence of such a term is not at all a common situation. We will show that the WZW term emerges quite naturally within the conventional derivation of the non-linear σ -model for disordered systems as a consequence of the non-analytical properties of the spectrum within the Brillouin zone. More specifically, the spin-current density in momentum space, $\mathbf{J}_{\mathbf{k}}$, in a d -wave superconductor is a 2×2 matrix in the Nambu spinor space. The WZW term arises just because the vector product $\mathbf{J}_{\mathbf{k}} \wedge \mathbf{J}_{\mathbf{k}}$ is finite and actually gives a measure of the vorticity around each node. In the presence of purely intra-node impurity scattering we obtain the same WZW action of the non-abelian bosonization from the 1d mapping^{19,20}, with however the inverse mean free path as a momentum upper cut-off instead of the inverse lattice spacing which is usual the Ultra-Violet-regularizer of the 1d Dirac theory. In this context we elucidate the role of the WZW term in providing the correct scaling behavior of the density of states depending on the impurity-potential range.

Another issue we clarify is the relationship between the coupling constant of the non-linear σ -model and the actual spin-conductance. The two quantities are known to coincide at the level of the Born approximation. However, rigorously speaking, the spin-conductance has to be calculated through a Kubo formula which involves advanced and retarded Green's functions. Since it makes a difference whether quasiparticles are right at the nodal points, with zero density of states, or slightly away from them, in which case the density of states is finite, one might wonder whether the two quantities, spin-conductance and non-linear σ -model coupling, remain equal even beyond the Born approximation, especially in the case of intra-node and inter-opposite-node scattering. We will show that this is actually the case.

Finally we will show that the range of the impurity potential crucially affects the energy scale at which localization precursor effects starts to appear. In particular we will explicitly show that, keeping fixed the inverse relaxation time within the Born approximation, $1/\tau_0$, and increasing the relative weight of the long-range with respect to the short-range components of the disorder, leads to strong reduction below $1/\tau_0$ of the energy scale for the on-set of localization. This may provide a natural explanation to the partial lack of experimental evidences of quasiparticle localization in superconducting cuprates.

The paper is organized as follows. In Section II we introduce the model which we reformulate within the fermionic path-integral formalism using the replica trick in Section III. The global symmetries of the action are discussed in Section IV. In Section V we start the derivation of the non-linear σ -model which includes two terms. The “conventional” one is worked out in Section VI, and its drawbacks discussed in Section VII. The “unconventional” WZW term is derived in Section VIII and its consequences for intra-nodal disorder are discussed in Section IX. In Section X we analyse the role of inter-nodal scattering processes and finally Section XI is devoted to the concluding remarks.

II. THE MODEL

The model we study is described by an Hartree-Fock Hamiltonian for d -wave superconductors in the presence of disorder

$$\begin{aligned}\mathcal{H} &= \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} [\epsilon_{\mathbf{k}} \tau_3 + \Delta_{\mathbf{k}} \tau_1] \psi_{\mathbf{k}} + \sum_{\mathbf{k}\mathbf{q}} V(\mathbf{q}) \psi_{\mathbf{k}}^{\dagger} \tau_3 \psi_{\mathbf{k}+\mathbf{q}} \\ &\equiv \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} H_{\mathbf{k}}^{(0)} \psi_{\mathbf{k}} + \sum_{\mathbf{k}\mathbf{q}} V(\mathbf{q}) \psi_{\mathbf{k}}^{\dagger} \tau_3 \psi_{\mathbf{k}+\mathbf{q}}.\end{aligned}\quad (1)$$

Here $V(\mathbf{q})$ is the impurity potential, $\psi_{\mathbf{k}}^{\dagger} = (c_{\mathbf{k}\uparrow}^{\dagger}, c_{-\mathbf{k}\downarrow}^{\dagger})$ the Nambu spinor, $\Delta_{\mathbf{k}} = \Delta (\cos k_x a - \cos k_y a)/2$ the superconducting gap with d -wave symmetry and $\epsilon_{\mathbf{k}}$ the band-dispersion measured with respect to the chemical potential. The Pauli matrices τ_i 's, $i = 1, 2, 3$, act on the Nambu spinor components. The spectrum of the Bogolubov quasiparticles is given by

$$E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}, \quad (2)$$

and shows four nodal points at $\mathbf{k}_1 = k_F (1, 1)/\sqrt{2}$, $\mathbf{k}_{\bar{1}} = -\mathbf{k}_1$, $\mathbf{k}_2 = k_F (-1, 1)/\sqrt{2}$ and $\mathbf{k}_{\bar{2}} = -\mathbf{k}_2$, k_F being the Fermi momentum along the diagonals of the Brillouin zone. Around

the nodes it is actually more convenient to rotate the axes by 45 degrees through

$$k_\xi = \frac{1}{\sqrt{2}}(k_x + k_y), \quad k_\eta = \frac{1}{\sqrt{2}}(k_y - k_x). \quad (3)$$

In the new reference frame, $\mathbf{k}_1 = k_F(1, 0)$ and $\mathbf{k}_2 = k_F(0, 1)$. In what follows we define

$$\boldsymbol{\kappa} \equiv \mathbf{k} - \mathbf{k}_a \quad (4)$$

the momentum deviation from any of the four nodal points \mathbf{k}_a , $a = 1, \bar{1}, 2, \bar{2}$. For small $|\boldsymbol{\kappa}| \ll k_F$ the spectrum around nodes $1(\bar{1})$ and $2(\bar{2})$ is, respectively,

$$E_{\mathbf{k}_1 + \boldsymbol{\kappa}} \equiv E_{1\boldsymbol{\kappa}} \simeq \sqrt{v_F^2 \kappa_\xi^2 + v_\Delta^2 \kappa_\eta^2}, \quad (5)$$

$$E_{\mathbf{k}_2 + \boldsymbol{\kappa}} \equiv E_{2\boldsymbol{\kappa}} \simeq \sqrt{v_F^2 \kappa_\eta^2 + v_\Delta^2 \kappa_\xi^2}, \quad (6)$$

thus having a conical Dirac-like form. Here $v_F = |\nabla \epsilon_{\mathbf{k}_a}|$ and $v_\Delta = |\nabla \Delta_{\mathbf{k}_a}|$.

In the presence of disorder the motion of the gapless quasiparticles may become diffusive in the hydrodynamic regime. However diffusive propagators appear only in those channels which refer to conserved quantities. Hence, in superconductors, only thermal and spin density fluctuations might acquire a diffusive behavior. Let us therefore briefly discuss some properties of the spin current operator which are going to play an important role in our analysis.

The z -component of the spin-current operator in the Nambu representation satisfies

$$\mathbf{q} \cdot \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \mathbf{J}_{\mathbf{k}}(\mathbf{q}) \psi_{\mathbf{k}+\mathbf{q}} = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \left[H_{\mathbf{k}+\mathbf{q}}^{(0)} - H_{\mathbf{k}}^{(0)} \right] \psi_{\mathbf{k}+\mathbf{q}}, \quad (7)$$

hence, for $\mathbf{q} \rightarrow 0$,

$$\mathbf{J}_{\mathbf{k}}(\mathbf{q}) \rightarrow \mathbf{J}_{\mathbf{k}} \equiv \nabla \epsilon_{\mathbf{k}} \tau_3 + \nabla \Delta_{\mathbf{k}} \tau_1. \quad (8)$$

Since the Pauli matrices anticommute, the following vector product turns out to be non-zero

$$\mathbf{J}_{\mathbf{k}} \wedge \mathbf{J}_{\mathbf{k}} = 2i \tau_2 \nabla \epsilon_{\mathbf{k}} \wedge \nabla \Delta_{\mathbf{k}}. \quad (9)$$

The vector product (9) actually probes the vorticity of the spectrum in momentum space; nodes “1” and “ $\bar{1}$ ” have the same vorticity, $\mathbf{J}_{\mathbf{k}} \wedge \mathbf{J}_{\mathbf{k}} \rightarrow 2i \tau_2 v_F v_\Delta$, as well as nodes “2” and “ $\bar{2}$ ”, although opposite of node “1”, $\mathbf{J}_{\mathbf{k}} \wedge \mathbf{J}_{\mathbf{k}} \rightarrow -2i \tau_2 v_F v_\Delta$. We are going to show that this property is crucial to uncover the physical behavior in the presence of disorder.

III. PATH INTEGRAL FORMULATION

To analyze the effect of disorder in this model, we are going to use a replica trick method within the fermionic path-integral approach. For that we associate to each fermionic operator Grassmann variables through

$$c_{\mathbf{k}\sigma}^\dagger \rightarrow \bar{c}_{\mathbf{k}\sigma}, \quad c_{\mathbf{k}\sigma} \rightarrow c_{\mathbf{k}\sigma}.$$

Notice that, unlike the original fermionic operators, $\bar{c}_{\mathbf{k}\sigma}$ and $c_{\mathbf{k}\sigma}$ are independent variables. After introducing the Nambu spinors

$$\bar{\psi}_{n\mathbf{k}} = (\bar{c}_{\mathbf{k}\uparrow}(i\omega_n), c_{\mathbf{k}\downarrow}(-i\omega_n)), \quad \psi_{n\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow}(i\omega_n) \\ \bar{c}_{\mathbf{k}\downarrow}(-i\omega_n) \end{pmatrix},$$

where $\bar{c}_{\mathbf{k}\sigma}(i\omega_n)$ and $c_{\mathbf{k}\sigma}(i\omega_n)$ are the Grassmann variables in Matsubara frequencies, the path-integral action without disorder reads

$$\mathcal{S}_0 = \sum_n \sum_{\mathbf{k}} \bar{\psi}_{n\mathbf{k}} [\epsilon_{\mathbf{k}} \tau_3 + \Delta_{\mathbf{k}} \tau_1 - i\omega_n] \psi_{n\mathbf{k}}. \quad (10)$$

The disorder introduces the additional term

$$\mathcal{S}_{imp} = \sum_n \sum_{\mathbf{k}\mathbf{q}} V(\mathbf{q}) \bar{\psi}_{n\mathbf{k}} \tau_3 \psi_{n\mathbf{k}+\mathbf{q}}. \quad (11)$$

Since $\bar{\psi}_{n\mathbf{k}}$ and $\psi_{n\mathbf{k}}$ are independent variables, the global transformation

$$\psi_{n\mathbf{k}} \rightarrow e^{i\theta\tau_2} \psi_{n\mathbf{k}}, \quad \bar{\psi}_{n\mathbf{k}} \rightarrow \bar{\psi}_{n\mathbf{k}} e^{i\theta\tau_2}, \quad (12)$$

becomes allowed within the path-integral and is indeed a symmetry transformation of the full action $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{imp}$ when $\omega_n = 0$. A finite frequency, $\omega_n \neq 0$, spoils this symmetry. If, in addition, the disorder is long range, namely it does not induce inter-node scatterings, it is possible to define four independent symmetry transformations of the above kind, one for each node. This type of *chiral* symmetry plays a crucial role for long range disorder, as was first emphasized in Ref. [19].

We notice that the disorder does not induce any mixing between different Matsubara frequencies, so that the action decouples into separate pieces, each one referring to a pair of opposite Matsubara frequencies, $\pm\omega_n$, which are coupled together by the superconducting term. For this reason we will just consider one of these pairs, with frequencies which we denote by $\pm\omega$, discarding all the others. This is enough to extract information about

the quasiparticle density of states (DOS) at given frequency as well as about transport coefficients.

In order to derive a long-wavelength effective action, we resort to the replica-trick technique, hence we first introduce N replicas of the Grassman variables, $\bar{c}_{\mathbf{k}\sigma}(\pm i\omega) \rightarrow \bar{c}_{a\mathbf{k}\sigma}(\pm i\omega)$ and $c_{\mathbf{k}\sigma}(\pm i\omega) \rightarrow c_{a\mathbf{k}\sigma}(\pm i\omega)$, with $a = 1, \dots, N$. At the end we shall send $N \rightarrow 0$. Next we define the column vectors $c_{\mathbf{k}}$ and $\bar{c}_{\mathbf{k}}$, with $4N$ elements (N replicas, 2 spins and 2 frequencies) $c_{a\mathbf{k}\sigma}(\pm i\omega)$ and $\bar{c}_{a\mathbf{k}\sigma}(\pm i\omega)$, respectively. Finally we introduce new Nambu spinors through³

$$\Psi_{\mathbf{k}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{c}_{-\mathbf{k}} \\ i\sigma_2 c_{\mathbf{k}} \end{pmatrix},$$

as well as

$$\bar{\Psi}_{\mathbf{k}} = \Psi_{-\mathbf{k}}^t c^t,$$

with the charge conjugacy operator defined by³

$$c = i \sigma_2 \tau_1.$$

Here the Pauli matrices σ_1 , σ_2 and σ_3 act on the spin components of the column vectors $c_{\mathbf{k}}$ and $\bar{c}_{\mathbf{k}}$, while τ_1 , τ_2 and τ_3 act on the Nambu components (the diagonal elements refer to the particle-hole channels and the off-diagonal ones to the particle-particle channels). For later convenience, we also introduce the Pauli matrices s_1 , s_2 and s_3 , which act on the opposite frequency partners, as well as the identity matrices in the spin, Nambu and frequency subspaces, σ_0 , τ_0 and s_0 , respectively.

By means of the above definitions, the clean action can be written as

$$\mathcal{S}_0 = \sum_{\mathbf{k}} \bar{\Psi}_{\mathbf{k}} [\epsilon_{\mathbf{k}} + i \Delta_{\mathbf{k}} \tau_2 s_1 - i\omega s_3] \Psi_{\mathbf{k}}. \quad (13)$$

Since we are interested in the low-energy long-wavelength behavior, we shall focus our attention only in small areas around each of the four nodal points. In the rotated reference frame (3), the nodes lie at $\mathbf{k}_1 = k_F (1, 0)$, $\mathbf{k}_{\bar{1}} = -\mathbf{k}_1$, $\mathbf{k}_2 = k_F (0, 1)$ and $\mathbf{k}_{\bar{2}} = -\mathbf{k}_2$. Using the definition (4) for the small momentum deviations away from the nodal points, we introduce new Grassmann variables defined around each node through

$$c_{a,\kappa} \equiv c_{\mathbf{k}_a + \kappa}, \quad \bar{c}_{a,\kappa} \equiv \bar{c}_{\mathbf{k}_a + \kappa},$$

$a = 1, \bar{1}, 2, \bar{2}$, as well as the corresponding Nambu spinors

$$\Psi_{1\kappa} = \begin{pmatrix} \bar{c}_{\bar{1}, -\kappa} \\ i\sigma_2 c_{1, \kappa} \end{pmatrix}, \Psi_{\bar{1}\kappa} = \begin{pmatrix} \bar{c}_{1, -\kappa} \\ i\sigma_2 c_{\bar{1}, \kappa} \end{pmatrix}, \Psi_{2\kappa} = \begin{pmatrix} \bar{c}_{\bar{2}, -\kappa} \\ i\sigma_2 c_{2, \kappa} \end{pmatrix}, \Psi_{\bar{2}\kappa} = \begin{pmatrix} \bar{c}_{2, -\kappa} \\ i\sigma_2 c_{\bar{2}, \kappa} \end{pmatrix}. \quad (14)$$

One notices that for $a = 1, 2$

$$\begin{aligned} \bar{\Psi}_{a\kappa} &= \frac{1}{\sqrt{2}} (-c_{\bar{a}, -\kappa}, -i \bar{c}_{a, \kappa} \sigma_2) = \Psi_{\bar{a}-\kappa}^t c^t, \\ \bar{\Psi}_{\bar{a}\kappa} &= \frac{1}{\sqrt{2}} (-c_{a, -\kappa}, -i \bar{c}_{\bar{a}, \kappa} \sigma_2) = \Psi_{a-\kappa}^t c^t. \end{aligned}$$

The non-disordered action expanded around the nodes reads

$$\begin{aligned} \mathcal{S}_0 &= \sum_{\kappa} \bar{\Psi}_{1\kappa} [v_F \kappa_\xi + i v_\Delta \kappa_\eta \tau_2 s_1] \Psi_{1\kappa} - \bar{\Psi}_{\bar{1}\kappa} [v_F \kappa_\xi + i v_\Delta \kappa_\eta \tau_2 s_1] \Psi_{\bar{1}\kappa} \\ &\quad + \bar{\Psi}_{2\kappa} [v_F \kappa_\eta + i v_\Delta \kappa_\xi \tau_2 s_1] \Psi_{2\kappa} - \bar{\Psi}_{\bar{2}\kappa} [v_F \kappa_\eta + i v_\Delta \kappa_\xi \tau_2 s_1] \Psi_{\bar{2}\kappa} \\ &\quad - i\omega \sum_{i=1, \bar{1}, 2, \bar{2}} \bar{\Psi}_{i\kappa} s_3 \Psi_{i\kappa}. \end{aligned}$$

We find useful to define new spinors with components in each of the nodes through

$$\Psi_{\kappa} = \begin{pmatrix} \Psi_{1\kappa} \\ \Psi_{2\kappa} \\ \Psi_{\bar{1}\kappa} \\ \Psi_{\bar{2}\kappa} \end{pmatrix},$$

so that

$$\bar{\Psi}_{\kappa} = (\Psi_{\bar{1}-\kappa}^t, \Psi_{\bar{2}-\kappa}^t, \Psi_{1-\kappa}^t, \Psi_{2-\kappa}^t) c^t = \Psi_{-\kappa}^t \gamma_1 c^t,$$

where the Pauli matrices γ 's act on the “ m ” and “ \bar{m} ” subspace, namely connect two opposite nodes. This naturally leads to a new charge conjugacy defined through

$$\mathcal{C} = c \gamma_1 = i \sigma_2 \tau_1 \gamma_1. \quad (15)$$

In what follows we denote as pair 1 the two opposite nodes “1” and “ $\bar{1}$ ”, and as pair 2 the other two nodes, “2” and “ $\bar{2}$ ”. Consequently we need to introduce also matrices in the two-pair subspace, 1 and 2, namely connecting adjacent nodes, which we will denote as ρ_0 , the identity, ρ_1 , ρ_2 and ρ_3 , the three Pauli matrices.

The clean action reads now

$$\begin{aligned} \mathcal{S}_0 = & \sum_{\boldsymbol{\kappa}} \bar{\Psi}_{\boldsymbol{\kappa}} \gamma_3 \frac{1}{2} (\kappa_{\xi} + \kappa_{\eta}) [v_F + i v_{\Delta} \tau_2 s_1] \Psi_{\boldsymbol{\kappa}} \\ & + \bar{\Psi}_{\boldsymbol{\kappa}} \gamma_3 \rho_3 \frac{1}{2} (\kappa_{\xi} - \kappa_{\eta}) [v_F - i v_{\Delta} \tau_2 s_1] \Psi_{\boldsymbol{\kappa}} - i\omega \sum_{\boldsymbol{\kappa}} \bar{\Psi}_{\boldsymbol{\kappa}} s_3 \Psi_{\boldsymbol{\kappa}} \end{aligned} \quad (16)$$

$$\equiv \sum_{\boldsymbol{\kappa}} \bar{\Psi}_{\boldsymbol{\kappa}} [\mathcal{H}_0(\boldsymbol{\kappa}) - i\omega s_3] \Psi_{\boldsymbol{\kappa}} \quad (17)$$

We notice that within this path-integral formulation, the four independent chiral symmetry transformations of the form (12) translate into

$$\Psi \rightarrow T \Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi} \mathcal{C} T^t \mathcal{C}^t,$$

with T given by any of the following expressions

$$T = e^{i\theta \tau_2 s_1 \gamma_0 \rho_0 \boldsymbol{\sigma} \cdot \mathbf{u}}, \quad (18)$$

$$T = e^{i\theta \tau_2 s_1 \gamma_3 \rho_0}, \quad (19)$$

$$T = e^{i\theta \tau_2 s_1 \gamma_0 \rho_3 \boldsymbol{\sigma} \cdot \mathbf{u}}, \quad (20)$$

$$T = e^{i\theta \tau_2 s_1 \gamma_3 \rho_3}, \quad (21)$$

where θ is a phase factor and \mathbf{u} an arbitrary unit vector.

We assume that the scattering potential $V(\mathbf{q}) = V(-\mathbf{q})^*$ induced by disorder has independent components which act inside each node, $V(\mathbf{q}) \equiv U_0(\mathbf{q})$, between opposite nodes, $V(\mathbf{q} + 2\mathbf{k}_1) \equiv U_1(\mathbf{q})$ and $V(\mathbf{q} + 2\mathbf{k}_2) \equiv U_2(\mathbf{q})$, and between adjacent nodes, $V(\mathbf{q} + \mathbf{k}_1 - \mathbf{k}_2) \equiv U_{12}(\mathbf{q})$ and $V(\mathbf{q} + \mathbf{k}_1 + \mathbf{k}_2) \equiv U_{1\bar{2}}(\mathbf{q})$, so that the impurity contribution to the action is

$$\begin{aligned} \mathcal{S}_{imp} = & \sum_{\boldsymbol{\kappa}, \mathbf{q}} U_0(\mathbf{q}) \bar{\Psi}_{\boldsymbol{\kappa}} \Psi_{\boldsymbol{\kappa}+\mathbf{q}} \\ & + \sum_{\boldsymbol{\kappa}, \mathbf{q}} U_1(\mathbf{q}) \bar{\Psi}_{1\boldsymbol{\kappa}} \Psi_{\bar{1}\boldsymbol{\kappa}+\mathbf{q}} + U_1(\mathbf{q})^* \bar{\Psi}_{\bar{1}\boldsymbol{\kappa}+\mathbf{q}} \Psi_{1\boldsymbol{\kappa}} \\ & + \sum_{\boldsymbol{\kappa}, \mathbf{q}} U_2(\mathbf{q}) \bar{\Psi}_{2\boldsymbol{\kappa}} \Psi_{\bar{2}\boldsymbol{\kappa}+\mathbf{q}} + U_2(\mathbf{q})^* \bar{\Psi}_{\bar{2}\boldsymbol{\kappa}+\mathbf{q}} \Psi_{2\boldsymbol{\kappa}} \\ & + \sum_{\boldsymbol{\kappa}, \mathbf{q}} U_{12}(\mathbf{q}) [\bar{\Psi}_{1\boldsymbol{\kappa}} \Psi_{2\boldsymbol{\kappa}+\mathbf{q}} + \bar{\Psi}_{\bar{2}\boldsymbol{\kappa}} \Psi_{\bar{1}\boldsymbol{\kappa}+\mathbf{q}}] + U_{12}(\mathbf{q})^* [\bar{\Psi}_{2\boldsymbol{\kappa}+\mathbf{q}} \Psi_{1\boldsymbol{\kappa}} + \bar{\Psi}_{\bar{1}\boldsymbol{\kappa}+\mathbf{q}} \Psi_{\bar{2}\boldsymbol{\kappa}}] \\ & + \sum_{\boldsymbol{\kappa}, \mathbf{q}} U_{1\bar{2}}(\mathbf{q}) [\bar{\Psi}_{1\boldsymbol{\kappa}} \Psi_{\bar{2}\boldsymbol{\kappa}+\mathbf{q}} + \bar{\Psi}_{2\boldsymbol{\kappa}} \Psi_{\bar{1}\boldsymbol{\kappa}+\mathbf{q}}] + U_{1\bar{2}}(\mathbf{q})^* [\bar{\Psi}_{\bar{2}\boldsymbol{\kappa}+\mathbf{q}} \Psi_{1\boldsymbol{\kappa}} + \bar{\Psi}_{\bar{1}\boldsymbol{\kappa}+\mathbf{q}} \Psi_{2\boldsymbol{\kappa}}]. \end{aligned} \quad (22)$$

Apart from the first term in the right hand side of (22), we have for convenience kept the node labels. Since the following relations hold:

$$\begin{aligned}\sum_{\kappa} \bar{\Psi}_{\bar{2}\kappa} \Psi_{\bar{1}\kappa+q} &= \sum_{\kappa} \Psi_{2-\kappa}^t c^t c^t \bar{\Psi}_{1-\kappa-q}^t = \sum_{\kappa} \bar{\Psi}_{1\kappa} \Psi_{2\kappa+q}, \\ \sum_{\kappa} \bar{\Psi}_{2\kappa} \Psi_{\bar{1}\kappa+q} &= \sum_{\kappa} \Psi_{\bar{2}-\kappa}^t c^t c^t \bar{\Psi}_{1-\kappa-q}^t = \sum_{\kappa} \bar{\Psi}_{1\kappa} \Psi_{\bar{2}\kappa+q}.\end{aligned}$$

we can rewrite (22) in the following way:

$$\begin{aligned}\mathcal{S}_{imp} &= \sum_{\kappa, q} U_0(\mathbf{q}) \bar{\Psi}_{\kappa} \Psi_{\kappa+q} \\ &+ \sum_{\kappa, q} U_1(\mathbf{q}) \bar{\Psi}_{1\kappa} \Psi_{\bar{1}\kappa+q} + U_1(\mathbf{q})^* \bar{\Psi}_{\bar{1}\kappa+q} \Psi_{1\kappa} \\ &+ \sum_{\kappa, q} U_2(\mathbf{q}) \bar{\Psi}_{2\kappa} \Psi_{\bar{2}\kappa+q} + U_2(\mathbf{q})^* \bar{\Psi}_{\bar{2}\kappa+q} \Psi_{2\kappa} \\ &+ 2 \sum_{\kappa, q} U_{12}(\mathbf{q}) \bar{\Psi}_{1\kappa} \Psi_{2\kappa+q} + U_{12}(\mathbf{q})^* \bar{\Psi}_{2\kappa+q} \Psi_{1\kappa} \\ &+ 2 \sum_{\kappa, q} U_{1\bar{2}}(\mathbf{q}) \bar{\Psi}_{1\kappa} \Psi_{\bar{2}\kappa+q} + U_{1\bar{2}}(\mathbf{q})^* \bar{\Psi}_{\bar{2}\kappa+q} \Psi_{1\kappa}.\end{aligned}\tag{23}$$

Finally we go back in real space, which now corresponds to the low-energy continuum limit of the original lattice model, and obtain the clean action

$$\begin{aligned}\mathcal{S}_0 &= \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) \frac{1}{2} \gamma_3 (-i\partial_\xi - i\partial_\eta) [v_F + i v_\Delta \tau_2 s_1] \Psi(\mathbf{r}) \\ &+ \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) \frac{1}{2} \gamma_3 \rho_3 (-i\partial_\xi + i\partial_\eta) [v_F - i v_\Delta \tau_2 s_1] \Psi(\mathbf{r}) \\ &- i\omega \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) s_3 \Psi(\mathbf{r}) \equiv \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) [\mathcal{H}_0(\mathbf{r}) - i\omega s_3] \Psi(\mathbf{r}),\end{aligned}\tag{24}$$

and the impurity term

$$\begin{aligned}\mathcal{S}_{imp} &= \int d\mathbf{r} U_0(\mathbf{r}) \bar{\Psi}(\mathbf{r}) \Psi(\mathbf{r}) \\ &+ U_1(\mathbf{r}) \bar{\Psi}_1(\mathbf{r}) \Psi_{\bar{1}}(\mathbf{r}) + U_1(\mathbf{r})^* \bar{\Psi}_{\bar{1}}(\mathbf{r}) \Psi_1(\mathbf{r}) \\ &+ U_2(\mathbf{r}) \bar{\Psi}_2(\mathbf{r}) \Psi_{\bar{2}}(\mathbf{r}) + U_2(\mathbf{r})^* \bar{\Psi}_{\bar{2}}(\mathbf{r}) \Psi_2(\mathbf{r}) \\ &+ 2 U_{12}(\mathbf{r}) \bar{\Psi}_1(\mathbf{r}) \Psi_2(\mathbf{r}) + 2 U_{12}(\mathbf{r})^* \bar{\Psi}_2(\mathbf{r}) \Psi_1(\mathbf{r}) \\ &+ 2 U_{1\bar{2}}(\mathbf{r}) \bar{\Psi}_1(\mathbf{r}) \Psi_{\bar{2}}(\mathbf{r}) + 2 U_{1\bar{2}}(\mathbf{r})^* \bar{\Psi}_{\bar{2}}(\mathbf{r}) \Psi_1(\mathbf{r}).\end{aligned}\tag{25}$$

Pauli matrices	subspace of action
$\sigma_0, \sigma_1, \sigma_2, \sigma_3$	spin components
$\tau_0, \tau_1, \tau_2, \tau_3$	Nambu components
s_0, s_1, s_2, s_3	opposite frequency partners
$\gamma_0, \gamma_1, \gamma_2, \gamma_3$	opposite nodal points
$\rho_0, \rho_1, \rho_2, \rho_3$	the two pairs of opposite nodes

TABLE I: The various two-components subspaces in which each set of Pauli matrices act

We further assume that the disorder is δ -like correlated with

$$\begin{aligned}
\langle U_0(\mathbf{r}) \rangle &= 0, \quad \langle U_0(\mathbf{r}) U_0(\mathbf{r}') \rangle = u^2 \delta(\mathbf{r} - \mathbf{r}'), \\
\langle U_1(\mathbf{r}) \rangle &= 0, \quad \langle U_1(\mathbf{r}) U_1(\mathbf{r}')^* \rangle = 2v^2 \delta(\mathbf{r} - \mathbf{r}'), \\
\langle U_2(\mathbf{r}) \rangle &= 0, \quad \langle U_2(\mathbf{r}) U_2(\mathbf{r}')^* \rangle = 2v^2 \delta(\mathbf{r} - \mathbf{r}'), \\
\langle U_{12}(\mathbf{r}) \rangle &= 0, \quad \langle U_{12}(\mathbf{r}) U_{12}(\mathbf{r}')^* \rangle = 2w^2 \delta(\mathbf{r} - \mathbf{r}'), \\
\langle U_{1\bar{2}}(\mathbf{r}) \rangle &= 0, \quad \langle U_{1\bar{2}}(\mathbf{r}) U_{1\bar{2}}(\mathbf{r}')^* \rangle = 2w^2 \delta(\mathbf{r} - \mathbf{r}').
\end{aligned} \tag{26}$$

We conclude by noticing that (i) if only $u \neq 0$ all transformations (18)–(21) leave the action invariant; (ii) if $u \neq 0$ and $v \neq 0$ only (18) and (20) are allowed; (iii) finally if $u \neq 0$, $v \neq 0$ and $w \neq 0$ only (18) remains.

IV. SYMMETRY PROPERTIES

Since we have been obliged to introduce so many Pauli matrices, including the identity denoted as the zeroth Pauli matrix, for sake of clarity we prefer to start this Section by first summarizing in Table I the subspaces in which any of them act.

Now, let us uncover all global symmetry transformations. We start assuming that only intra-node disorder is present, namely $U_0 \neq 0$ but $U_1 = U_2 = U_{12} = U_{1\bar{2}} = 0$.

If the frequency is zero, the action $S_0 + S_{imp}$ is invariant under unitary global transfor-

mations T such that

$$\begin{aligned}
\mathcal{C}^t T^t \mathcal{C} \gamma_3 T &= \gamma_3, \\
\mathcal{C}^t T^t \mathcal{C} \gamma_3 \tau_2 s_1 T &= \gamma_3 \tau_2 s_1, \\
\mathcal{C}^t T^t \mathcal{C} \gamma_3 \rho_3 T &= \gamma_3 \rho_3, \\
\mathcal{C}^t T^t \mathcal{C} \gamma_3 \rho_3 \tau_2 s_1 T &= \gamma_3 \rho_3 \tau_2 s_1, \\
\mathcal{C}^t T^t \mathcal{C} T &= 1.
\end{aligned}$$

They imply that

$$\mathcal{C}^t T^t \mathcal{C} = T^{-1}, \quad (27)$$

as well as that

$$[T, \gamma_3] = 0, \quad [T, \rho_3] = 0, \quad [T, \tau_2 s_1] = 0. \quad (28)$$

We parametrize T as

$$T = \frac{1}{2} (T_1 + T_2) \rho_0 + \frac{1}{2} (T_1 - T_2) \rho_3, \quad (29)$$

where the suffix “1” refers to the pair 1 (opposite nodes 1 and $\bar{1}$) and “2” to the pair 2 (opposite nodes 2 and $\bar{2}$). The symmetry modes for pair 1 and 2 can be parametrized by the $16N \times 16N$ unitary operators

$$T_{1(2)} = \frac{1}{2} (\gamma_0 + \gamma_3) V_{1(2)} + \frac{1}{2} (\gamma_0 - \gamma_3) V_{\bar{1}(\bar{2})} \equiv \begin{pmatrix} V_{1(2)} & 0 \\ 0 & V_{\bar{1}(\bar{2})} \end{pmatrix},$$

where the V ’s are $8N \times 8N$ unitary matrices. Because of (27) we get

$$\mathcal{C}^t T_{1(2)}^t \mathcal{C} = T_{1(2)}^{-1} = T_{1(2)}^\dagger,$$

and $V_{1(2)}$ and $V_{\bar{1}(\bar{2})}$ are actually not independent as

$$c^t V_{1(2)}^t c = V_{\bar{1}(\bar{2})}^\dagger. \quad (30)$$

Therefore

$$T_{1(2)} = \begin{pmatrix} V_{1(2)} & 0 \\ 0 & c^t \left(V_{1(2)}^\dagger \right)^t c \end{pmatrix},$$

is indeed parametrized by a single $8N \times 8N$ unitary matrix $V_{1(2)}$. Since the two pairs of nodes behave similarly, in what follows we drop the suffices 1 and 2. According to (28) we still need to impose that

$$[V, \tau_2 s_1] = 0. \quad (31)$$

To this purpose let us introduce the following unitary operation

$$U = \left[\frac{1}{2} (\tau_0 + \tau_2) - \frac{1}{2} (\tau_0 - \tau_2) s_3 \right] e^{-i\frac{\pi}{4} s_2}. \quad (32)$$

which transforms

$$\Psi \rightarrow U \Psi, \quad \bar{\Psi} \rightarrow \Psi^t U^t c^t = \Psi^t c^t U^\dagger = \bar{\Psi} U^\dagger,$$

so that for any operator \mathcal{O}

$$\bar{\Psi} \mathcal{O} \Psi \rightarrow \bar{\Psi} U^\dagger \mathcal{O} U \Psi.$$

One readily shows that

$$U^\dagger \tau_2 s_1 U = s_3, \quad U^\dagger s_3 U = -s_1, \quad (33)$$

hence (31) transforms into

$$[V, s_3] = 0, \quad (34)$$

which is fulfilled by the general expression

$$V = \frac{1}{2} (A + B) s_0 + \frac{1}{2} (A - B) s_3, \quad (35)$$

with A and B being independent $4N \times 4N$ unitary matrices. Therefore the original symmetry turns out to be $U(4N) \times U(4N)$ for pair 1, and analogously for pair 2, in total $U(4N) \times U(4N) \times U(4N) \times U(4N)$.

We notice that, in the presence of a finite frequency, $\omega \neq 0$, we shall further impose through (33) that

$$[V, s_1] = 0,$$

which is satisfied by

$$V = C s_0,$$

with C belonging to $U(4N)$. Therefore the symmetry is lowered by the frequency down to $U(4N)$ for the pair 1 times $U(4N)$ for the pair 2.

Within the non-linear σ -model approach to disordered systems, the frequency plays actually the role of a symmetry-breaking field. This leads to the identification of the transverse modes V^\perp as those satisfying

$$s_1 V^\perp s_1 = (V^\perp)^{-1} = (V^\perp)^\dagger,$$

implying $A = B^\dagger$ in Eq.(35) so that we can write

$$V^\perp = \frac{1}{2} \left[\sqrt{G} + \sqrt{G^\dagger} \right] s_0 + \frac{1}{2} \left[\sqrt{G} - \sqrt{G^\dagger} \right] s_3, \quad (36)$$

where G belongs to $U(4N)$. In other words the coset space spanned by the transverse modes is still a group, namely $U(4N)$. It is convenient to factorize out of G the abelian component. For that we write

$$G = e^{i\sqrt{\frac{\pi}{N}}\phi} g, \quad (37)$$

where ϕ is a scalar and g belongs to $SU(4N)$. The forms (36) and (37) will be useful in the following to express the non-linear σ -model directly in terms of g and ϕ . After the transformation (32) from Eq.(30) we find that

$$V_{\overline{1(2)}}^\perp = \left[\tau_1 \sigma_2 (V_{1(2)}^\perp)^\dagger \tau_1 \sigma_2 \right]^t = \left[\tau_1 \sigma_2 s_1 V_{1(2)}^\perp s_1 \tau_1 \sigma_2 \right]^t, \quad (38)$$

which leads to

$$\phi_{\overline{1(2)}} = -\phi_{1(2)}, \quad g_{\overline{1(2)}} = \tau_1 \sigma_2 \left(g_{1(2)}^\dagger \right)^t \sigma_2 \tau_1. \quad (39)$$

Let us conclude by showing what would change for more general disorder potential. First let us consider the case in which the disorder also contains terms which couple opposite nodes, i.e. $U_1 \neq 0$ and $U_2 \neq 0$. In this case the γ_3 -modes are not anymore allowed by symmetry, so we have to impose that $V_{1(2)} = V_{\overline{1(2)}}$, namely, through (39), that $\phi_{1(2)} = 0$ and

$$g_{1(2)}^\dagger = g_{1(2)}^{-1} = \tau_1 \sigma_2 g_{1(2)}^t \sigma_2 \tau_1. \quad (40)$$

The coset becomes now the group $Sp(2N)$ of unitary-symplectic matrices (also called $USp(4N)$) for pair 1 and analogously for pair 2, i.e. $Sp(2N) \times Sp(2N)$.

Finally, if also scattering between adjacent nodes is allowed, $U_{12} \neq 0$ and $U_{\overline{12}} \neq 0$, then also the ρ_3 modes do not leave the action invariant. In this case we have to impose that $V_1^\perp = V_2^\perp$, so that the coset space is the group $Sp(2N)$.

V. THE NON-LINEAR σ -MODEL

In what follows we begin analyzing just the case in which the disorder only induces intra-node scattering processes. At the end we will return back to the most general case. Within

the replica trick, we can average the action over the disorder probability distribution, after which (25) with $v = w = 0$ [see Eq. (26)] transforms into

$$\mathcal{S}_{imp} = -\frac{u^2}{2} \int d\mathbf{r} [\bar{\Psi}(\mathbf{r})\Psi(\mathbf{r})]^2. \quad (41)$$

We define $32N \times 32N$ matrices $X(\mathbf{r})$ by

$$X(\mathbf{r}) \equiv \Psi(\mathbf{r}) \bar{\Psi}(\mathbf{r}),$$

so that (41) can be also written as

$$\mathcal{S}_{imp} = \frac{u^2}{2} \int d\mathbf{r} \text{Tr} (X(\mathbf{r}) X(\mathbf{r})). \quad (42)$$

By means of an Hubbard-Stratonovich transformation one can show that

$$\exp \left\{ -\frac{u^2}{2} \int d\mathbf{r} \text{Tr} (X(\mathbf{r}) X(\mathbf{r})) \right\} = \int \mathcal{D}Q(\mathbf{r}) \exp \left\{ -\int d\mathbf{r} \frac{1}{2u^2} \text{Tr} [Q(\mathbf{r})^2] - i \text{Tr} [Q(\mathbf{r}) X(\mathbf{r})] \right\}, \quad (43)$$

with $Q(\mathbf{r})$ being hermitian $32N \times 32N$ auxiliary matrix fields.

In conclusion the full action, (24) plus (43), becomes

$$\begin{aligned} \mathcal{S} = & \frac{1}{2u^2} \int d\mathbf{r} \text{Tr} [Q(\mathbf{r})^2] \\ & + \int d\mathbf{r} \bar{\Psi}(\mathbf{r}) [\mathcal{H}_0(\mathbf{r}) - i Q(\mathbf{r}) - i\omega s_3] \Psi(\mathbf{r}). \end{aligned} \quad (44)$$

One obtains the effective action which describes the auxiliary field $Q(\mathbf{r})$ by integrating out the Grassmann variables, thus getting

$$\mathcal{S}[Q] = \frac{1}{2u^2} \int d\mathbf{r} \text{Tr} [Q(\mathbf{r})^2] - \frac{1}{2} \text{Tr} \ln [iQ + i\omega s_3 - \mathcal{H}_0]. \quad (45)$$

We further proceed in the derivation of a long-wavelength effective action for $Q(\mathbf{r})$ by following the conventional approach. First of all we calculate the saddle point expression of $Q(\mathbf{r})$, which we denote by $Q_0 s_3$, assuming it is uniform, in the presence of an infinitesimal symmetry breaking term ωs_3 . Then we neglect longitudinal fluctuations and parametrize the actual $Q(\mathbf{r})$ by

$$Q(\mathbf{r}) \simeq T(\mathbf{r})^{-1} Q_0 s_3 T(\mathbf{r}) = \frac{1}{2} (\rho_0 + \rho_3) Q_1(\mathbf{r}) + \frac{1}{2} (\rho_0 - \rho_3) Q_2(\mathbf{r}), \quad (46)$$

where through (29) $Q_{ab}(\mathbf{r}) = \delta_{ab} Q_a(\mathbf{r})$ with

$$Q_a(\mathbf{r}) = T_a(\mathbf{r})^{-1} Q_0 s_3 T_a(\mathbf{r}) = Q_0 s_3 T_a(\mathbf{r})^2 = T_a(\mathbf{r})^{-2} Q_0 s_3, \quad (47)$$

being the auxiliary field in the pair subspace $a, b = 1, 2$. Notice that, even though the most general $Q(\mathbf{r})$ would couple the nodes together, namely would include off-diagonal elements $Q_{12}(\mathbf{r})$, (46) does not contain any mixing term. This simply reflects that the off-diagonal components are not diffusive, hence massive.

In order to derive the long-wavelength action we find it convenient to decompose the action into the real part

$$\begin{aligned} \mathcal{S}_{NL\sigma M} = & \frac{1}{2u^2} \int d\mathbf{r} \text{Tr} [Q(\mathbf{r})^2] \\ & - \frac{1}{4} \text{Tr} \ln [iQ + i\omega s_3 - \mathcal{H}_0] - \frac{1}{4} \text{Tr} \ln [-iQ - i\omega s_3 - \mathcal{H}_0^\dagger], \end{aligned} \quad (48)$$

which, as we shall see, gives the conventional non-linear σ -model, and the imaginary part

$$\mathcal{S}_\Gamma = -\frac{1}{4} \text{Tr} \ln [iQ + i\omega s_3 - \mathcal{H}_0] + \frac{1}{4} \text{Tr} \ln [-iQ - i\omega s_3 - \mathcal{H}_0^\dagger], \quad (49)$$

which we will show gives rise to a WZW term.

VI. CONVENTIONAL σ -MODEL

By means of (46), the real part of the action, (48), can be written as

$$\mathcal{S}_{NL\sigma M} = \frac{V_{\text{eff}}}{2u^2} 32N Q_0^2 - \frac{1}{4} \text{Tr} \ln [\mathcal{G}^{-1}], \quad (50)$$

where V_{eff} is the effective volume corresponding to the long-wavelength theory (roughly speaking $V_{\text{eff}} = V/4$, since we have implicitly folded the Brillouin zone into a single quadrant, in order to make all nodes coincide), and

$$\begin{aligned} \mathcal{G}^{-1} &= \mathcal{H}_0 \mathcal{H}_0^\dagger + Q_0^2 + \omega^2 + \omega (Q s_3 + s_3 Q) + i\mathcal{H}_0 Q - iQ \mathcal{H}_0^\dagger \\ &\equiv \mathcal{G}_0^{-1} - \Sigma, \end{aligned} \quad (51)$$

where

$$\begin{aligned} \mathcal{G}_0^{-1} &= \mathcal{H}_0 \mathcal{H}_0^\dagger + Q_0^2 + \omega^2, \\ \Sigma &= -\omega \{Q, s_3\} - i\mathcal{H}_0 Q + iQ \mathcal{H}_0^\dagger. \end{aligned}$$

We notice that $s_3 \mathcal{H}_0 s_3 = \mathcal{H}_0^\dagger$, therefore

$$i\mathcal{H}_0 Q - iQ \mathcal{H}_0^\dagger = i\mathcal{H}_0 Q s_3 s_3 - iQ s_3 \mathcal{H}_0 s_3 = i[\mathcal{H}_0, Q s_3] s_3 = \mathbf{J} \cdot \nabla Q,$$

where we made use of the equivalence $Q s_3 \equiv T^{-2}$ and of the expression of the spin current operator in the long wave-length limit

$$\mathbf{J}(\mathbf{r}) = i [\mathcal{H}_0(\mathbf{r}), \mathbf{r}]. \quad (52)$$

Analogously one can show that

$$i \mathcal{H}_0 Q - i Q \mathcal{H}_0^\dagger = i s_3 \mathcal{H}_0^\dagger s_3 Q - i s_3 s_3 Q \mathcal{H}_0^\dagger = -i s_3 [s_3 Q, \mathcal{H}_0^\dagger] = \nabla Q \cdot \mathbf{J}^\dagger,$$

so that the self-energy operator can be written as

$$\Sigma = -\omega \{Q, s_3\} - \mathbf{J} \cdot \nabla Q = -\omega \{Q, s_3\} - \nabla Q \cdot \mathbf{J}^\dagger. \quad (53)$$

We then expand the action up to second order in Σ and obtain

$$\begin{aligned} \mathcal{S}_{NL\sigma M} &= \frac{V_{\text{eff}}}{2u^2} 32N Q_0^2 - \frac{1}{4} \text{Tr} \ln [\mathcal{G}^{-1}] \\ &= \frac{V_{\text{eff}}}{2u^2} 32N Q_0^2 + \frac{1}{4} \text{Tr} \ln [\mathcal{G}_0] - \frac{1}{4} \text{Tr} \ln [1 - \mathcal{G}_0 \Sigma] \\ &= \frac{V_{\text{eff}}}{2u^2} 32N Q_0^2 + \frac{1}{4} \text{Tr} \ln [\mathcal{G}_0] + \frac{1}{4} \text{Tr} [\mathcal{G}_0 \Sigma] + \frac{1}{8} \text{Tr} [\mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma] \\ &= \frac{V_{\text{eff}}}{2u^2} 32N Q_0^2 + \frac{1}{4} \text{Tr} \ln [\mathcal{G}_0] - \frac{\omega}{2} \text{Tr} [Q s_3 \mathcal{G}_0] \\ &\quad + \frac{\omega^2}{8} \text{Tr} [\mathcal{G}_0 \{Q, s_3\} \mathcal{G}_0 \{Q, s_3\}] + \frac{1}{8} \text{Tr} [\mathcal{G}_0 \vec{\nabla} Q \cdot \mathbf{J}^\dagger \mathcal{G}_0 \mathbf{J} \cdot \vec{\nabla} Q]. \end{aligned} \quad (54)$$

Notice that we have arrived to $\mathcal{S}_{NL\sigma M}$ in terms of $\vec{\nabla} Q$ without passing through a gauge transformation to carry out the gradient expansion so avoiding any problem related to a proper treatment of the Jacobian determinant²⁰.

A. Saddle point equation

In momentum space one finds that

$$\begin{aligned} \mathcal{H}_{0\kappa} \mathcal{H}_{0\kappa}^\dagger &= \frac{1}{2} (\kappa_\xi^2 + \kappa_\eta^2) (v_F^2 + v_\Delta^2) + \rho_3 \frac{1}{2} (\kappa_\xi^2 - \kappa_\eta^2) (v_F^2 - v_\Delta^2) \\ &= \frac{1}{2} (E_{1\kappa}^2 + E_{2\kappa}^2) + \rho_3 \frac{1}{2} (E_{1\kappa}^2 - E_{2\kappa}^2), \end{aligned}$$

where $E_{1\kappa}$ is the spectrum of quasiparticles around nodes “1” and “ $\bar{1}$ ”, and analogously for $E_{2\kappa}$, see (5) and (6). Therefore

$$\mathcal{G}_{0\kappa} = \frac{1}{2} [\mathcal{G}_{1\kappa} + \mathcal{G}_{2\kappa}] + \rho_3 \frac{1}{2} [\mathcal{G}_{1\kappa} - \mathcal{G}_{2\kappa}],$$

where

$$\mathcal{G}_{1\kappa} = \frac{1}{E_{1\kappa}^2 + Q_0^2 + \omega^2}, \quad \mathcal{G}_{2\kappa} = \frac{1}{E_{2\kappa}^2 + Q_0^2 + \omega^2},$$

so that

$$\mathcal{G}_{0\kappa}^2 = \frac{1}{2} [\mathcal{G}_{1\kappa}^2 + \mathcal{G}_{2\kappa}^2] + \rho_3 \frac{1}{2} [\mathcal{G}_{1\kappa}^2 - \mathcal{G}_{2\kappa}^2] \quad (55)$$

The saddle-point equation is obtained by taking $Q(\mathbf{r}) = Q_0$ and minimizing the action, which leads to the self-consistency equation

$$\frac{Q_0}{Q_0 + \omega} = u^2 \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}^2 + (Q_0 + \omega)^2}, \quad (56)$$

originally derived in Ref. [1]. The solution of this equation, in the limit $\omega \rightarrow 0$, reads

$$Q_0 = \Lambda \exp\left(-\frac{\pi v_F v_\Delta}{2u^2}\right),$$

where Λ is an ultraviolet cut-off which is roughly the energy scale above which the spectrum deviates appreciably from a linear one, in other words $\Lambda \simeq \Delta$. We notice that in the generic case with finite inter-nodal scattering, namely with non zero v^2 and w^2 , the saddle point equation remains the same apart from $u^2 \rightarrow u^2 + 2w^2 + v^2$.

The quasiparticle density of states $N(\epsilon)$, after the analytic continuation of $i\omega$ to the positive real axis, $i\omega \rightarrow \epsilon > 0$, turns out to be, for small ϵ ,

$$N(\epsilon) = \frac{2}{\pi^2 v_F v_\Delta} \left[Q_0 \ln \frac{\Lambda}{\sqrt{\epsilon^2 + Q_0^2}} + \frac{\epsilon}{2} \left(\frac{\pi}{2} - \tan^{-1} \frac{Q_0^2 - \epsilon^2}{2\epsilon Q_0} \right) \right]. \quad (57)$$

Therefore the density of states at the saddle-point acquires a finite value $N_0 \sim Q_0/u^2$ at the chemical potential $\epsilon = 0$, while, for $\epsilon \gg Q_0$, turns back to the linear dependence $N(\epsilon) \sim \epsilon$ as in the absence of disorder.

B. Frequency dependent terms

The first order term in the frequency is just

$$-\frac{\omega}{2} \text{Tr} [Q s_3 \mathcal{G}_0] = -\omega \frac{\pi}{2} \frac{N_0}{Q_0} \int d\mathbf{r} \text{Tr} [Q(\mathbf{r}) s_3]. \quad (58)$$

The second order term is readily found to be

$$\frac{\omega^2}{8} \text{Tr} [\mathcal{G}_0 \{Q, s_3\} \mathcal{G}_0 \{Q, s_3\}] = \frac{1}{32\pi^2} \frac{\omega^2}{v_F v_\Delta} \frac{1}{Q_0^2} \int d\mathbf{r} \text{Tr} [\{Q(\mathbf{r}), s_3\}^2]. \quad (59)$$

This term is negligible as compared to (58) for frequencies ω much smaller than $v_F v_\Delta N_0$. The latter is therefore the energy scale below which diffusion sets in at the mean field level, namely the so-called inverse relaxation time in the Born approximation, $1/\tau_0$.

C. Gradient expansion

The Fourier transform of the current operator is

$$\mathbf{J}(\mathbf{q}) = \gamma_3 \frac{1}{2} [\mathbf{J}_1(\mathbf{q}) + \mathbf{J}_2(\mathbf{q})] + \gamma_3 \rho_3 \frac{1}{2} [\mathbf{J}_1(\mathbf{q}) - \mathbf{J}_2(\mathbf{q})], \quad (60)$$

where, at long wavelengths $\mathbf{q} \rightarrow 0$,

$$\begin{aligned} \mathbf{J}_1(\mathbf{q}) &\rightarrow \mathbf{J}_1 = (v_F, i v_\Delta \tau_2 s_1) \\ \mathbf{J}_2(\mathbf{q}) &\rightarrow \mathbf{J}_2 = (i v_\Delta \tau_2 s_1, v_F). \end{aligned}$$

It is then easy to derive the following expression of the tensor product $\mathbf{J} \otimes \mathbf{J}^\dagger$

$$\begin{aligned} \mathbf{J} \otimes \mathbf{J}^\dagger &= \begin{pmatrix} J_\xi J_\xi^\dagger & J_\xi J_\eta^\dagger \\ J_\eta J_\xi^\dagger & J_\eta J_\eta^\dagger \end{pmatrix} = \frac{1}{2} (v_F^2 + v_\Delta^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \frac{1}{2} \rho_3 (v_F^2 - v_\Delta^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i v_F v_\Delta \tau_2 s_1 \rho_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \quad (61)$$

To evaluate the second order gradient correction in (54) we notice that

$$\begin{aligned} \frac{1}{8} \text{Tr} \left[\mathcal{G}_0 \vec{\nabla} Q \cdot \mathbf{J}^\dagger \mathcal{G}_0 \mathbf{J} \cdot \vec{\nabla} Q \right] &= \frac{1}{8} \sum_{i,j} \text{Tr} \left[\partial_i Q \partial_j Q \mathcal{G}_0 J_j^\dagger \mathcal{G}_0 J_i \right] \\ &= \frac{1}{16} \sum_{i,j} \text{Tr} \left[\partial_i Q \partial_j Q \left(\mathcal{G}_0 J_j^\dagger \mathcal{G}_0 J_i + \mathcal{G}_0 J_i^\dagger \mathcal{G}_0 J_j \right) \right] \end{aligned}$$

In the long wavelength limit

$$\begin{aligned} \mathcal{G}_0 J_j^\dagger \mathcal{G}_0 J_i + \mathcal{G}_0 J_i^\dagger \mathcal{G}_0 J_j &= \frac{1}{V} \sum_{\kappa} \mathcal{G}_{0\kappa} J_j^\dagger \mathcal{G}_{0\kappa} J_i + \mathcal{G}_{0\kappa} J_i^\dagger \mathcal{G}_{0\kappa} J_j \\ &= \delta_{ij} \frac{1}{4V} \sum_{\kappa} [\mathcal{G}_{1\kappa}^2 + \mathcal{G}_{2\kappa}^2] [(v_F^2 + v_\Delta^2) + \rho_3 (v_F^2 - v_\Delta^2) (\delta_{i\xi} - \delta_{i\eta})], \end{aligned}$$

since $\sum_{\boldsymbol{\kappa}} \mathcal{G}_{1\boldsymbol{\kappa}}^2 - \mathcal{G}_{2\boldsymbol{\kappa}}^2 = 0$. We find that

$$\begin{aligned} \frac{1}{4V} \sum_{\boldsymbol{\kappa}} \mathcal{G}_{1\boldsymbol{\kappa}}^2 + \mathcal{G}_{2\boldsymbol{\kappa}}^2 &= \frac{1}{2V} \sum_{\boldsymbol{\kappa}} \mathcal{G}_{1\boldsymbol{\kappa}}^2 \\ &= \frac{1}{2} \int \frac{d\boldsymbol{\kappa}_\xi d\boldsymbol{\kappa}_\eta}{4\pi^2} \left(\frac{1}{v_F^2 \boldsymbol{\kappa}_\xi^2 + v_\Delta^2 \boldsymbol{\kappa}_\eta^2 + Q_0^2} \right)^2 \\ &= \frac{1}{8\pi v_F v_\Delta} \int_0^\infty dx^2 \left(\frac{1}{x^2 + Q_0^2} \right)^2 = \frac{1}{8\pi v_F v_\Delta} \frac{1}{Q_0^2}. \end{aligned}$$

The Drude spin-conductivity is defined by¹

$$\sigma = \frac{1}{4\pi^2} \frac{v_F^2 + v_\Delta^2}{v_F v_\Delta}, \quad (62)$$

so that the second order gradient correction can be written as

$$\frac{\pi}{16 Q_0^2} \sigma \int d\mathbf{r} \text{Tr} [\boldsymbol{\nabla} Q(\mathbf{r}) \cdot \boldsymbol{\nabla} Q(\mathbf{r})] + \frac{v_F^2 - v_\Delta^2}{v_F^2 + v_\Delta^2} \text{Tr} \{ \rho_3 [\partial_\xi Q(\mathbf{r}) \partial_\xi Q(\mathbf{r}) - \partial_\eta Q(\mathbf{r}) \partial_\eta Q(\mathbf{r})] \}.$$

D. The final form of $S_{NL\sigma M}$

Collecting all contributions which are second order in the gradient expansion and first order in the frequency we eventually obtain

$$\begin{aligned} S_{NL\sigma M} &= \frac{\pi}{16 Q_0^2} \sigma \int d\mathbf{r} \text{Tr} [\boldsymbol{\nabla} Q(\mathbf{r}) \cdot \boldsymbol{\nabla} Q(\mathbf{r})] \\ &\quad + \frac{\pi}{16 Q_0^2} \sigma \int d\mathbf{r} \frac{v_F^2 - v_\Delta^2}{v_F^2 + v_\Delta^2} \text{Tr} \{ \rho_3 [\partial_\xi Q(\mathbf{r}) \partial_\xi Q(\mathbf{r}) - \partial_\eta Q(\mathbf{r}) \partial_\eta Q(\mathbf{r})] \} \\ &\quad - \omega \frac{\pi N_0}{2 Q_0} \int d\mathbf{r} \text{Tr} [Q(\mathbf{r}) s_3] \\ &= \sum_{i=1,2} \int d\mathbf{r} \frac{\pi}{16 Q_0^2} \sigma \text{Tr} [\partial_\mu Q_i(\mathbf{r}) \alpha_{\mu\nu}^{(i)} \partial_\nu Q_i(\mathbf{r})] - \omega \frac{\pi N_0}{2 Q_0} \text{Tr} [Q_i(\mathbf{r}) s_3]. \quad (63) \end{aligned}$$

Here we have defined a metric tensor for the pair 1 which includes nodes “1” and “ $\bar{1}$ ”,

$$\alpha^{(1)} = \frac{2}{v_F^2 + v_\Delta^2} \begin{pmatrix} v_F^2 & 0 \\ 0 & v_\Delta^2 \end{pmatrix},$$

and for the pair 2, i.e. for nodes “2” and “ $\bar{2}$ ”,

$$\alpha^{(2)} = \frac{2}{v_F^2 + v_\Delta^2} \begin{pmatrix} v_\Delta^2 & 0 \\ 0 & v_F^2 \end{pmatrix}.$$

In reality the above action is not the most general one allowed by symmetry. As we discussed earlier, the theory possesses two chiral abelian sectors, which actually occur in the singlet channels $\gamma_3 \tau_2 s_1 \rho_0$ and $\gamma_3 \tau_2 s_1 \rho_3$, see (19) and (21). In analogy with Ref. [24], we expect that upon integrating out longitudinal fluctuations the following term would appear:

$$\delta \mathcal{S}_{NL\sigma M} = \frac{\pi}{16^2 Q_0^4} \sigma \Pi \sum_{i=1,2} \int d\mathbf{r} \text{Tr} [\gamma_3 \tau_2 s_1 Q_i(\mathbf{r}) \partial_\mu Q_i(\mathbf{r})] \alpha_{\mu\nu}^{(i)} \text{Tr} [\gamma_3 \tau_2 s_1 Q_i(\mathbf{r}) \partial_\nu Q_i(\mathbf{r})], \quad (64)$$

with $\Pi \propto u^2$.

In conclusion the most general non-linear σ model is given by

$$\begin{aligned} \mathcal{S}_{NL\sigma M} = & \sum_{i=1,2} \int d\mathbf{r} \frac{\pi}{16 Q_0^2} \sigma \text{Tr} [\partial_\mu Q_i(\mathbf{r}) \alpha_{\mu\nu}^{(i)} \partial_\nu Q_i(\mathbf{r})] - \omega \frac{\pi N_0}{2 Q_0} \text{Tr} [Q_i(\mathbf{r}) s_3] \\ & + \frac{\pi}{16^2 Q_0^4} \sigma \Pi \text{Tr} [\gamma_3 \tau_2 s_1 Q_i(\mathbf{r}) \partial_\mu Q_i(\mathbf{r})] \alpha_{\mu\nu}^{(i)} \text{Tr} [\gamma_3 \tau_2 s_1 Q_i(\mathbf{r}) \partial_\nu Q_i(\mathbf{r})]. \end{aligned} \quad (65)$$

VII. FAILURE OF THE CONVENTIONAL σ -MODEL

The non-linear σ -model (65) belongs to one of the known chiral σ -models encountered when two-sublattice symmetry holds, see Refs. [9,24,25]. If we simply borrow known results, see e.g. Table I in Ref. [9], we should expect that the β -function of the conductivity vanishes in the zero replica limit, $N \rightarrow 0$. That would imply absence of localization and persistence of diffusive modes. Moreover we should predict a quasiparticle density of states $N(\epsilon)$ which diverges like^{26,27,28}

$$N(\epsilon) \simeq \frac{1}{\epsilon} \exp \left[-A \sqrt[3]{\ln \frac{B}{\epsilon}} \right], \quad (66)$$

with A and B some positive constants. This is clearly suspicious since in the absence of disorder the density of states actually vanishes, $N(\epsilon) \sim \epsilon$.

One is tempted to correlate the above suspicious result with what is found for the elementary loops of the Wilson-Polyakov renormalization group approach. Here one integrates out iteratively degrees of freedom in a momentum shell from the highest cut-off Λ to Λ/s , with $s > 0$ eventually sent to infinity. In our case these fundamental loops for either nodes are given by

$$\frac{1}{\pi\sigma} \int_{\Lambda/s < |\boldsymbol{\kappa}| < \Lambda} \frac{d\boldsymbol{\kappa}}{4\pi^2} \frac{1}{\kappa_\mu \alpha_{\mu\nu}^{(1,2)} \kappa_\nu} = \ln s \equiv g \ln s,$$

and provide the definition of the dimensionless coupling constant g , which should necessarily be small to justify a loop-expansion in $g \ln s \ll 1$. In our case it turns out that $g = 1$, making any loop expansion meaningless.

This results is at odds with the standard non linear σ -models for disordered systems where $g \sim 1/\sigma \ll 1$ for weak disorder. Here, whatever weak is the disorder, yet $g = 1$. This peculiar fact was originally discussed in Ref. [19], where the authors identified correctly the complete failure of the non-crossing approximation as a starting point of perturbation theory due to the absence of small control parameters. This might lead to the conclusion that the non-linear σ -model we have so far derived is useless in this problem, since it heavily relies upon the assumption that quantum fluctuations around the saddle-point solution can be controlled perturbatively.

In the following Section we are going to show that the above conclusion is not correct and that one only needs to be more careful in deriving the proper non-linear σ -model action.

VIII. WESS-ZUMINO-NOVIKOV-WITTEN TERM

Indeed we have not yet accomplished all our plan, as we still need to evaluate the imaginary part of the action (49). At leading order we can drop the frequency dependence of (49), hence

$$\mathcal{S}_\Gamma = -\frac{1}{4}\text{Tr} \ln [iQ - \mathcal{H}_0] + \frac{1}{4}\text{Tr} \ln [-iQ - \mathcal{H}_0^\dagger]. \quad (67)$$

It is more convenient to evaluate the variation of \mathcal{S}_Γ along a massless path defined through

$$\delta Q(\mathbf{r}) Q(\mathbf{r}) + Q(\mathbf{r}) \delta Q(\mathbf{r}) = 0. \quad (68)$$

We find that

$$\delta \mathcal{S}_\Gamma = -\frac{i}{4}\text{Tr} [G \delta Q] + \frac{i}{4}\text{Tr} [G^\dagger \delta Q], \quad (69)$$

where

$$G = (iQ - \mathcal{H}_0)^{-1}. \quad (70)$$

We notice that, in the long wavelength limit,

$$\left(-iQ - \mathcal{H}_0^\dagger\right) (iQ - \mathcal{H}_0) \simeq \mathcal{G}_0^{-1} - \nabla Q \cdot \mathbf{J} = \mathcal{G}_0^{-1} - \mathbf{J}^\dagger \cdot \nabla Q, \quad (71)$$

and

$$(iQ - \mathcal{H}_0) \left(-iQ - \mathcal{H}_0^\dagger\right) \simeq \mathcal{G}_0^{-1} + \nabla Q \cdot \mathbf{J} = \mathcal{G}_0^{-1} + \mathbf{J}^\dagger \cdot \nabla Q, \quad (72)$$

hence

$$\begin{aligned}
G &= [1 - \mathcal{G}_0 \nabla Q \cdot \mathbf{J}]^{-1} \mathcal{G}_0 \left(-iQ - \mathcal{H}_0^\dagger \right) \\
&\simeq [1 + \mathcal{G}_0 \nabla Q \cdot \mathbf{J} + \mathcal{G}_0 \nabla Q \cdot \mathbf{J} \mathcal{G}_0 \nabla Q \cdot \mathbf{J}] \mathcal{G}_0 \left(-iQ - \mathcal{H}_0^\dagger \right), \\
G^\dagger &= [1 + \mathcal{G}_0 \nabla Q \cdot \mathbf{J}]^{-1} \mathcal{G}_0 (iQ - \mathcal{H}_0) \\
&\simeq [1 - \mathcal{G}_0 \nabla Q \cdot \mathbf{J} + \mathcal{G}_0 \nabla Q \cdot \mathbf{J} \mathcal{G}_0 \nabla Q \cdot \mathbf{J}] \mathcal{G}_0 (iQ - \mathcal{H}_0).
\end{aligned}$$

The leading non-vanishing contribution to (69) derives from the second order gradient correction to $G - G^\dagger$, which reads

$$G - G^\dagger \simeq -2i \mathcal{G}_0 \nabla Q \cdot \mathbf{J} \mathcal{G}_0 \nabla Q \cdot \mathbf{J} \mathcal{G}_0 Q. \quad (73)$$

Therefore

$$\begin{aligned}
\delta S_\Gamma &= -\frac{1}{2} \text{Tr} [\delta Q \mathcal{G}_0 \nabla Q \cdot \mathbf{J} \mathcal{G}_0 \nabla Q \cdot \mathbf{J} \mathcal{G}_0 Q] \\
&= -\frac{1}{2} \text{Tr} [\delta Q \mathcal{G}_0 \nabla Q \cdot \mathbf{J} \mathcal{G}_0 \mathbf{J}^\dagger \cdot \nabla Q \mathcal{G}_0 Q]
\end{aligned} \quad (74)$$

The only term which contributes comes from the anti-symmetric component of the tensor $\mathbf{J} \otimes \mathbf{J}^\dagger$:

$$\frac{1}{2} \left(J_i J_j^\dagger - J_j J_i^\dagger \right) = -i \epsilon_{ij} v_F v_\Delta \tau_2 s_1 \rho_3, \quad (75)$$

being ϵ_{ij} the Levi-Civita tensor, thus leading to

$$\delta S_\Gamma = i \Gamma \sum_{i,j} \epsilon_{i,j} \text{Tr} [\partial_j Q Q \delta Q \partial_i Q \tau_2 s_1 \rho_3] \quad (76)$$

where

$$\begin{aligned}
\Gamma &= \frac{1}{4V} \sum_{\kappa} v_F v_\Delta [\mathcal{G}_{1\kappa}^3 + \mathcal{G}_{2\kappa}^3] \\
&= \frac{1}{8\pi} \int_0^\infty dx^2 \left(\frac{1}{x^2 + Q_0^2} \right)^3 = \frac{1}{16\pi} \frac{1}{Q_0^4}.
\end{aligned} \quad (77)$$

Introducing a fictitious coordinate which parametrizes the massless path, we finally get

$$\begin{aligned}
\mathcal{S}_\Gamma &= i \frac{1}{12\pi} \frac{1}{4Q_0^6} \int d^3\mathbf{r} \epsilon_{\mu\nu\eta} \text{Tr} [\tau_2 s_1 \rho_3 Q(\mathbf{r}) \partial_\mu Q(\mathbf{r}) Q(\mathbf{r}) \partial_\nu Q(\mathbf{r}) Q(\mathbf{r}) \partial_\eta Q(\mathbf{r})] \\
&= i \frac{1}{12\pi} \frac{1}{4Q_0^6} \int d^3\mathbf{r} \epsilon_{\mu\nu\eta} \text{Tr} [\tau_2 s_1 Q_1(\mathbf{r}) \partial_\mu Q_1(\mathbf{r}) Q_1(\mathbf{r}) \partial_\nu Q_1(\mathbf{r}) Q_1(\mathbf{r}) \partial_\eta Q_1(\mathbf{r})] \\
&\quad - i \frac{1}{12\pi} \frac{1}{4Q_0^6} \int d^3\mathbf{r} \epsilon_{\mu\nu\eta} \text{Tr} [\tau_2 s_1 Q_2(\mathbf{r}) \partial_\mu Q_2(\mathbf{r}) Q_2(\mathbf{r}) \partial_\nu Q_2(\mathbf{r}) Q_2(\mathbf{r}) \partial_\eta Q_2(\mathbf{r})].
\end{aligned} \quad (78)$$

This actually represents opposite WZW terms for each of the two pairs of nodes, namely for each of the two independent $U(1) \times SU(4N)$ σ -models. It is clear that if the disorder coupled the two pairs of nodes, we would be forced to identify $Q_1(\mathbf{r})$ with $Q_2(\mathbf{r})$, so that the WZW term would cancel in that case. To make (78) more explicit we remind that, for $a = 1, 2$,

$$Q_a(\mathbf{r}) = T_a(\mathbf{r})^{-2} s_3 Q_0 = s_3 Q_0 T_a(\mathbf{r})^2, \quad (79)$$

where

$$T_a^2(\mathbf{r}) = \frac{1}{2} (\gamma_0 + \gamma_3) V_a^2(\mathbf{r}) + \frac{1}{2} (\gamma_0 - \gamma_3) c^t [V_a^{-2}(\mathbf{r})]^t c, \quad (80)$$

and

$$V_a^2(\mathbf{r}) = U \left[\frac{1}{2} (s_0 + s_3) e^{i\sqrt{\frac{\pi}{N}} \phi_a(\mathbf{r})} g_a(\mathbf{r}) + \frac{1}{2} (s_0 - s_3) e^{-i\sqrt{\frac{\pi}{N}} \phi_a(\mathbf{r})} g_a(\mathbf{r})^\dagger \right] U^\dagger, \quad (81)$$

U being defined in Eq. (32). By means of (79), (80) and (81), the expression (78) can be finally written as

$$\begin{aligned} \mathcal{S}_\Gamma = & i \frac{1}{12\pi} \int d^3\mathbf{r} \epsilon_{\mu\nu\eta} \text{Tr} [g_1(\mathbf{r})^\dagger \partial_\mu g_1(\mathbf{r}) g_1(\mathbf{r})^\dagger \partial_\nu g_1(\mathbf{r}) g_1(\mathbf{r})^\dagger \partial_\eta g_1(\mathbf{r})] \\ & - i \frac{1}{12\pi} \int d^3\mathbf{r} \epsilon_{\mu\nu\eta} \text{Tr} [g_2(\mathbf{r})^\dagger \partial_\mu g_2(\mathbf{r}) g_2(\mathbf{r})^\dagger \partial_\nu g_2(\mathbf{r}) g_2(\mathbf{r})^\dagger \partial_\eta g_2(\mathbf{r})], \end{aligned} \quad (82)$$

appropriate for two WZW models $SU(4N)_1$.

Let us express all other terms in the action by means of ϕ_a and g_a . First of all the density of state operator becomes

$$\frac{1}{Q_0} \text{Tr} [s_3 Q(\mathbf{r})] = \text{Tr} [T^2(\mathbf{r})] = 2 \sum_{a=1}^2 e^{i\sqrt{\frac{\pi}{N}} \phi_a(\mathbf{r})} \text{Tr} [g_a(\mathbf{r})] + e^{-i\sqrt{\frac{\pi}{N}} \phi_a(\mathbf{r})} \text{Tr} [g_a(\mathbf{r})^\dagger]. \quad (83)$$

Moreover one readily finds that

$$\frac{1}{Q_0^2} \text{Tr} [\partial_\mu Q_a(\mathbf{r}) \alpha_{\mu\nu}^{(a)} \partial_\nu Q_a(\mathbf{r})] = 16\pi \partial_\mu \phi_a(\mathbf{r}) \alpha_{\mu\nu}^{(a)} \partial_\mu \phi_a(\mathbf{r}) + 4 \text{Tr} [\partial_\mu g_a(\mathbf{r}) \alpha_{\mu\nu}^{(a)} \partial_\nu g_a(\mathbf{r})^\dagger],$$

and

$$\frac{1}{Q_0^4} \text{Tr} [\gamma_3 \tau_2 s_1 Q_a(\mathbf{r}) \partial_\mu Q_a(\mathbf{r})] \alpha_{\mu\nu}^{(a)} \text{Tr} [\gamma_3 \tau_2 s_1 Q_a(\mathbf{r}) \partial_\nu Q_a(\mathbf{r})] = \frac{\pi (16N)^2}{N} \partial_\mu \phi_a(\mathbf{r}) \alpha_{\mu\nu} \partial_\nu \phi_a(\mathbf{r}).$$

In conclusion the action expressed in terms of the fundamental fields and including the WZW term reads

$$\begin{aligned} \mathcal{S}_{WZW} = & \sum_{a=1,2} \int d\mathbf{r} \frac{\pi}{4} \sigma \text{Tr} [\partial_\mu g_a(\mathbf{r}) \alpha_{\mu\nu}^{(a)} \partial_\nu g_a(\mathbf{r})^\dagger] + \pi^2 \sigma (1 + N\Pi) \partial_\mu \phi_a(\mathbf{r}) \alpha_{\mu\nu}^{(a)} \partial_\nu \phi_a(\mathbf{r}) \\ & - \omega \pi N_0 \left\{ e^i \sqrt{\frac{\pi}{N}} \phi_a(\mathbf{r}) \text{Tr} [g_a(\mathbf{r})] + e^{-i} \sqrt{\frac{\pi}{N}} \phi_a(\mathbf{r}) \text{Tr} [g_a(\mathbf{r})^\dagger] \right\} + S_\Gamma^{(a)}, \end{aligned} \quad (84)$$

with

$$S_\Gamma^{(a)} = \pm i \frac{1}{12\pi} \int d^3\mathbf{r} \epsilon_{\mu\nu\eta} \text{Tr} [g_a(\mathbf{r})^\dagger \partial_\mu g_a(\mathbf{r}) g_a(\mathbf{r})^\dagger \partial_\nu g_a(\mathbf{r}) g_a(\mathbf{r})^\dagger \partial_\eta g_a(\mathbf{r})], \quad (85)$$

where the plus refers to $a = 1$ and the minus to $a = 2$.

IX. CONSEQUENCES OF THE WZW TERM

We have just shown that the actual field theory which describes d -wave superconductors in the presence of a disorder which only permits intra-node scattering processes is not a conventional non linear σ -model but instead it represents two decoupled $U(1) \times SU(4N)_1$ WZW models. Moreover if, for instance, we consider pair 1, then

$$\begin{aligned} & \frac{\pi}{4} \sigma \frac{2}{v_F^2 + v_\Delta^2} \int d^2\mathbf{r} v_F^2 \text{Tr} [\partial_\xi g_a(\mathbf{r}) \partial_\xi g_a(\mathbf{r})^\dagger] + v_\Delta^2 \text{Tr} [\partial_\eta g_a(\mathbf{r}) \partial_\eta g_a(\mathbf{r})^\dagger] \\ & = \frac{\pi}{4} \frac{1}{4\pi^2} \frac{v_F^2 + v_\Delta^2}{v_F v_\Delta} \frac{2}{v_F^2 + v_\Delta^2} \int d^2\mathbf{r} v_F^2 \text{Tr} [\partial_\xi g_a(\mathbf{r}) \partial_\xi g_a(\mathbf{r})^\dagger] + v_\Delta^2 \text{Tr} [\partial_\eta g_a(\mathbf{r}) \partial_\eta g_a(\mathbf{r})^\dagger] \\ & = \frac{1}{8\pi} \int \frac{d^2\mathbf{r}}{v_F v_\Delta} v_F^2 \text{Tr} [\partial_\xi g_a(\mathbf{r}) \partial_\xi g_a(\mathbf{r})^\dagger] + v_\Delta^2 \text{Tr} [\partial_\eta g_a(\mathbf{r}) \partial_\eta g_a(\mathbf{r})^\dagger], \end{aligned}$$

which, upon the change of variable $\xi \rightarrow v_F \xi$ and $\eta \rightarrow v_\Delta \eta$, shows that each WZW model is right at its fixed point. Hence there is no ambiguity in the zero replica limit.

Now we can draw some consequences of what we have found. The first is that the average value of the density of states (DOS) at the chemical potential stays zero, as in the absence of disorder and contrary to the Born approximation. In particular the dimension of the density of states operator in the zero replica limit $N \rightarrow 0$ is

$$\Delta_Q = \frac{N-1}{N} + \frac{1}{N(1+N\Pi)} \rightarrow 1 - \Pi,$$

while the dimension of the frequency is $[\omega] = 2 - (1 - \Pi) = 1 + \Pi$. This implies that the DOS at finite frequency behaves as

$$N(\omega) \sim |\omega|^{\frac{1-\Pi}{1+\Pi}}, \quad (86)$$

in agreement with Ref. [19].

Notice the (86) stems from the fact that the WZW terms modifies the non-linear σ -model results leading to Eq. (66) in two ways: i) it makes Π unrenormalized as well as σ , ii) it adds the further contribution one to the dimension of the density of states operator. The only difference between the 1d mapping and the non-linear σ -model results for the density of states are the constant factors fixed by the range of validity. 1d mapping: $N(\omega) \sim 1/(v_F v_\Delta) \omega (\omega/\Delta)^{-2\Pi/(1+\Pi)}$ with $\omega < \Delta$; non-linear σ -model: $N(\omega) \sim N_0 (\omega \tau_0)^{(1-\Pi)/(1+\Pi)}$ with $\omega < 1/\tau_0$, N_0 being the saddle point value of the density of states. At leading order in u^2 the matching of the two expressions is provided by $N_0 \tau_0 \sim 1$.

The second consequence concerns the transport properties. We have shown that within the Born approximation a finite spin-conductivity arises. Is this result still true beyond that approximation? The renormalization group analysis which identifies the spin conductivity with the coupling of the non-linear σ -model would say that σ_s stays unrenormalized to its fixed point value. Can we understand this in the 1d mapping language?

In order to answer this question, we introduce a uniform spin vector potential within the action, which in our path-integral approach has the form

$$\mathbf{A} = \mathbf{A}_0 \sigma_3 + \mathbf{A}_1 \sigma_3 s_1. \quad (87)$$

The action becomes now a functional of \mathbf{A} , i.e. $\mathcal{S} \rightarrow \mathcal{S}(\mathbf{A})$, and the spin-conductivity turns out to be given by

$$\sigma_s = \frac{1}{2\pi} \left(\frac{\partial^2 \ln \mathcal{Z}(\mathbf{A})}{\partial A_{0,\xi}^2} - \frac{\partial^2 \ln \mathcal{Z}(\mathbf{A})}{\partial A_{1,\xi}^2} \right)_{\mathbf{A}=0} = \frac{1}{2\pi} \left(\frac{\partial^2 \ln \mathcal{Z}(\mathbf{A})}{\partial A_{0,\eta}^2} - \frac{\partial^2 \ln \mathcal{Z}(\mathbf{A})}{\partial A_{1,\eta}^2} \right)_{\mathbf{A}=0}, \quad (88)$$

where

$$\mathcal{Z}(\mathbf{A}) = \int DQ(\mathbf{r})^{-\mathcal{S}(\mathbf{A})}$$

is the generating functional in the presence of \mathbf{A} . It is possible to show that at second order in \mathbf{A} one needs just to make the following substitution in the action

$$\partial_\mu Q(\mathbf{r}) \rightarrow \partial_\mu Q(\mathbf{r}) + \frac{i}{2} [Q(\mathbf{r}), A_\mu]. \quad (89)$$

Upon the action of (32)

$$\mathbf{A} \rightarrow U^\dagger \mathbf{A} U = \mathbf{A}_0 \sigma_3 + \mathbf{A}_1 \sigma_3 \tau_2 s_3,$$

and we can show that (89) implies the following transformation law of the matrix field $G(\mathbf{r})$ in the presence of \mathbf{A}

$$G(\mathbf{r}) \rightarrow e^{-\frac{i}{2} \int d\mathbf{r}' \cdot (\mathbf{A}_0(\mathbf{r}') \sigma_3 - \mathbf{A}_1(\mathbf{r}') \tau_2 \sigma_3)} G(\mathbf{r}) e^{\frac{i}{2} \int d\mathbf{r}' \cdot (\mathbf{A}_0(\mathbf{r}') \sigma_3 + \mathbf{A}_1(\mathbf{r}') \tau_2 \sigma_3)}. \quad (90)$$

In order to better understand the role of \mathbf{A} it is convenient to translate the WZW action into the language of the underneath free one-dimensional fermions. One may identify the component $G_{1;\alpha\beta}(\mathbf{r})$ of the matrix field for pair “1”, where α and β run over $4N$ indices, with

$$G_{1;\alpha\beta}(\mathbf{r}) \rightarrow i \Psi_{1;R\alpha}(\mathbf{r}) \Psi_{1;L\beta}^\dagger(\mathbf{r}), \quad (91)$$

being $\Psi_{1;\alpha R}(\mathbf{r})$ and $\Psi_{1;\alpha L}(\mathbf{r})$ right and left moving one dimensional Fermi fields, respectively, and the two component vector \mathbf{r} playing the role of space and time coordinates. Since pair “2” has the opposite WZW term of pair “1”, it is more appropriate to define

$$G_{2;\alpha\beta}(\mathbf{r}) \rightarrow -i \Psi_{2;L\alpha}(\mathbf{r}) \Psi_{2;R\beta}^\dagger(\mathbf{r}), \quad (92)$$

which formally yields to equal WZW terms. Then (90) implies for the fermions the transformation laws

$$\begin{aligned} \Psi_{1;R}(\mathbf{r}) &\rightarrow e^{-\frac{i}{2} \int d\mathbf{r}' \cdot (\mathbf{A}_0(\mathbf{r}') \sigma_3 - \mathbf{A}_1(\mathbf{r}') \tau_2 \sigma_3)} \Psi_{1;R}(\mathbf{r}), \\ \Psi_{1;L}(\mathbf{r}) &\rightarrow e^{-\frac{i}{2} \int d\mathbf{r}' \cdot (\mathbf{A}_0(\mathbf{r}') \sigma_3 + \mathbf{A}_1(\mathbf{r}') \tau_2 \sigma_3)} \Psi_{1;L}(\mathbf{r}), \\ \Psi_{2;R}(\mathbf{r}) &\rightarrow e^{-\frac{i}{2} \int d\mathbf{r}' \cdot (\mathbf{A}_0(\mathbf{r}') \sigma_3 + \mathbf{A}_1(\mathbf{r}') \tau_2 \sigma_3)} \Psi_{2;R}(\mathbf{r}), \\ \Psi_{2;L}(\mathbf{r}) &\rightarrow e^{-\frac{i}{2} \int d\mathbf{r}' \cdot (\mathbf{A}_0(\mathbf{r}') \sigma_3 - \mathbf{A}_1(\mathbf{r}') \tau_2 \sigma_3)} \Psi_{2;L}(\mathbf{r}), \end{aligned}$$

which we may interpret as if \mathbf{A}_0 couples to the spin-density operator

$$\sum_{i=1,2} \Psi_{i;R}^\dagger(\mathbf{r}) \sigma_3 \Psi_{i;R}(\mathbf{r}) + \Psi_{i;L}^\dagger(\mathbf{r}) \sigma_3 \Psi_{i;L}(\mathbf{r}), \quad (93)$$

while \mathbf{A}_1 to the spin-current operator

$$\Psi_{1;R}^\dagger(\mathbf{r}) \sigma_3 \tau_2 \Psi_{1;R}(\mathbf{r}) - \Psi_{1;L}^\dagger(\mathbf{r}) \sigma_3 \tau_2 \Psi_{1;L}(\mathbf{r}) - \Psi_{2;R}^\dagger(\mathbf{r}) \sigma_3 \tau_2 \Psi_{2;R}(\mathbf{r}) + \Psi_{2;L}^\dagger(\mathbf{r}) \sigma_3 \tau_2 \Psi_{2;L}(\mathbf{r}). \quad (94)$$

Since the fermions are free, apart from the abelian sector which is not coupled to \mathbf{A} , the susceptibility towards \mathbf{A}_0 is the same as that towards \mathbf{A}_1 , hence the spin-conductivity would seem to vanish at ω strictly equal to zero, again unlike what we found within the Born

approximation. Actually one has to be more careful in drawing such a conclusion. Let us suppose to do the same calculation at finite frequency ω and afterwards send $\omega \rightarrow 0$. A frequency $\omega \neq 0$ plays the role of an explicit dimerization in the one dimensional fermionic problem:

$$- \pi \omega N_0 \sum_{i=1,2} \text{Tr} \left[G_i(\mathbf{r}) + G_i^\dagger(\mathbf{r}) \right] = -i \pi \omega N_0 \sum_{i,\alpha} \Psi_{i;R\alpha}^\dagger(\mathbf{r}) \Psi_{i;L\alpha}(\mathbf{r}) - \Psi_{i;L\alpha}^\dagger(\mathbf{r}) \Psi_{i;R\alpha}(\mathbf{r}). \quad (95)$$

It is straightforward to show that in the presence of a finite dimerization the current-current susceptibility is finite and practically equal to that one in the absence of dimerization, while the density-density susceptibility is zero. This discontinuous behavior at $\omega = 0$ as opposed to $\omega \rightarrow 0$ is again a manifestation of the chiral anomaly which plays such an important role in this problem¹⁹. Since it is physically more appropriate to identify the spin conductance through the $\omega \rightarrow 0$ limit, we conclude that, in spite of the vanishing DOS, the spin conductivity is finite. In other words, in spite of the fact that the DOS is vanishing at the chemical potential, namely that quasiparticle motion is undamped hence remains ballistic, yet the spin conductivity acquires a finite value in agreement with the Drude approximation.

X. INTER-NODE SCATTERING PROCESSES

So far we have just considered the role of impurity scattering within each node. Now let us extend our analysis by including also inter-node scattering processes. Upon integrating out the most general disorder, we find two additional terms. The first describes opposite node scattering processes, and it reads:

$$\delta \mathcal{S}_{imp}^I = -v^2 \int d\mathbf{r} \left[\overline{\Psi}_1(\mathbf{r}) \Psi_{\overline{1}}(\mathbf{r}) \right] \left[\overline{\Psi}_{\overline{1}}(\mathbf{r}) \Psi_1(\mathbf{r}) \right] + \left[\overline{\Psi}_2(\mathbf{r}) \Psi_{\overline{2}}(\mathbf{r}) \right] \left[\overline{\Psi}_{\overline{2}}(\mathbf{r}) \Psi_2(\mathbf{r}) \right]. \quad (96)$$

The second derives from the impurity scattering processes which couple the two pairs of nodes, and it is given by

$$\begin{aligned} \delta \mathcal{S}_{imp}^{II} = & -w^2 \int d\mathbf{r} \left[\overline{\Psi}_1(\mathbf{r}) \Psi_2(\mathbf{r}) \right] \left[\overline{\Psi}_2(\mathbf{r}) \Psi_1(\mathbf{r}) \right] + \left[\overline{\Psi}_{\overline{1}}(\mathbf{r}) \Psi_{\overline{2}}(\mathbf{r}) \right] \left[\overline{\Psi}_{\overline{2}}(\mathbf{r}) \Psi_{\overline{1}}(\mathbf{r}) \right] \\ & + \left[\overline{\Psi}_1(\mathbf{r}) \Psi_{\overline{2}}(\mathbf{r}) \right] \left[\overline{\Psi}_{\overline{2}}(\mathbf{r}) \Psi_1(\mathbf{r}) \right] + \left[\overline{\Psi}_{\overline{1}}(\mathbf{r}) \Psi_2(\mathbf{r}) \right] \left[\overline{\Psi}_2(\mathbf{r}) \Psi_{\overline{1}}(\mathbf{r}) \right]. \end{aligned} \quad (97)$$

As before we can decouple the four-fermion terms by introducing auxiliary Hubbard-Stratonovich fields. Since these fields are expected to be massive, we can further expand the

action up to second order in those fields after integrating out fermions. We finally integrate on the auxiliary massive fields. The net result is still an action for the Q matrices only which includes now the additional terms:

$$\delta\mathcal{S}_{imp}^I = -\lambda^I \frac{N_0}{Q_0} \int d\mathbf{r} \operatorname{Tr} [Q(\mathbf{r}) \gamma_1 Q(\mathbf{r}) \gamma_1], \quad (98)$$

and

$$\delta\mathcal{S}_{imp}^{II} = -\lambda^{II} \frac{N_0}{Q_0} \int d\mathbf{r} \sum_{i=0,1} \operatorname{Tr} [Q(\mathbf{r}) \rho_1 \gamma_i Q(\mathbf{r}) \rho_1 \gamma_i], \quad (99)$$

where

$$\lambda^I \sim \frac{v^2}{u^2}, \quad \lambda^{II} \sim \frac{w^2}{u^2}.$$

The first term (98) tends to lock $G_1 = G_1$ and $G_2 = G_2$, while (99) locks $G_1 = G_2$.

When opposite node scattering is added, still leaving pairs of opposite nodes uncoupled, only the symmetric Q -combinations of opposite nodes stay massless, the Π -term disappears and the coset for soft modes is $\operatorname{Sp}(2N)$ for each pair of nodes. The β -function is vanishing only because of the contribution of the WZW term, and density of states vanishes with a universal exponent in agreement with the known results^{7,19,22}. In the absence of isotropic scattering the vanishing of the β function still indicates that the spin and heat conductivities have a metallic behavior.

Finally, in the generic case in which all nodes are coupled only the four nodes symmetric combination of the Q 's is required for the soft modes. The coset is again $\operatorname{Sp}(2N)$, but now it represents degrees of freedom coming from all nodes. The two WZW terms are written in terms of the same Q and they cancel since they have opposite sign. The action then reduce to the $S_{NL\sigma M}$ as derived by [5].

A. Scaling analysis of the general model

In order to elucidate the role of inter-node scattering processes, it is convenient to transform the 2 dimensional non-linear σ -model into a 1+1 dimensional model of interacting fermions.

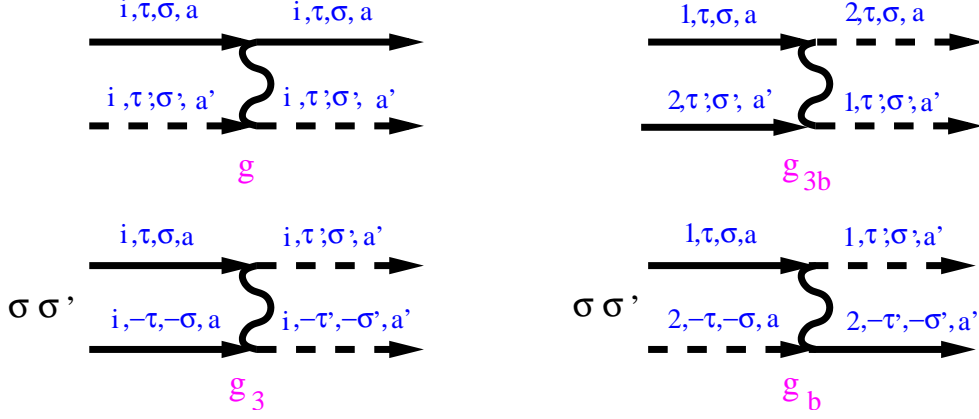


FIG. 1: Interaction vertices for the most general disorder. Solid(dashed) lines refer to right(left) moving fermions. The label $i = 1, 2$ refers to the two pairs of opposite nodes, $\tau, \tau' = \pm 1$ to the two Nambu components, $\sigma, \sigma' = \uparrow, \downarrow$ to the spin and $a, a' = 1, \dots, N$ to the replicas. The symbol $\sigma \sigma'$ in front of the g_3 and g_{3b} coupling constants means $+1$ if $\sigma = \sigma'$ and -1 otherwise.

We represent the matrix fields for pair “1” according to:

$$\begin{aligned}
G_{1;\tau\sigma a, \tau'\sigma' a'} &= i \Psi_{1;R,\tau\sigma a} \Psi_{1;L,\tau'\sigma' a'}^\dagger, \\
G_{1;\tau\sigma a, \tau'\sigma' a'}^\dagger &= -i \Psi_{1;L,\tau\sigma a} \Psi_{1;R,\tau'\sigma' a'}^\dagger, \\
G_{\bar{1};\tau\sigma a, \tau'\sigma' a'} &= \left(\tau_1 \sigma_2 G_1^\dagger \sigma_2 \tau_1 \right)_{\tau'\sigma' a', \tau\sigma a} \\
&= -i \sigma \sigma' \Psi_{1;L,-\tau'-\sigma' a'} \Psi_{1;R,-\tau-\sigma a}^\dagger, \\
G_{\bar{1};\tau\sigma a, \tau'\sigma' a'}^\dagger &= i \sigma \sigma' \Psi_{1;R,-\tau'-\sigma' a'} \Psi_{1;L,-\tau-\sigma a}^\dagger.
\end{aligned} \tag{100}$$

As we mentioned, since pair “2” has the opposite WZW term of pair “1”, it is convenient to define $G_{2;\tau\sigma a, \tau'\sigma' a'} = -i \Psi_{2;L,\tau\sigma a} \Psi_{2;R,\tau'\sigma' a'}^\dagger$. In such a way the two WZW terms become equal leaving no ambiguity in mapping the non-linear σ -model onto a one-dimensional model of interacting electrons with the interaction vertices drawn in Fig. 1. We notice that the coupling g in Fig. 1 derives from the two terms in Eq. (64). The bare values of the coupling constants are approximately

$$\begin{aligned}
g^{(0)} &\simeq u^2, \\
g_3^{(0)} &\simeq v^2, \\
g_b^{(0)} &= g_{3b}^{(0)} \simeq w^2.
\end{aligned}$$

Generally $u^2 \geq w^2 \geq v^2$, the equality holding only for short range impurity potential. An important observation is that two-loop corrections to the renormalization group (RG)

equations vanish in the zero replica limit, hence the RG equations valid up to two-loops are found to be:

$$\begin{aligned}\frac{dg}{d \ln s} &= g_3^2 + g_b^2 + g_{3b}^2, \\ \frac{dg_3}{d \ln s} &= 4 g g_3 + 4 g_b g_{3b}, \\ \frac{dg_b}{d \ln s} &= 2 g g_b + 2 g_3 g_{3b}, \\ \frac{dg_{3b}}{d \ln s} &= 2 g_3 g_b + 2 g_{3b} g,\end{aligned}$$

where $s \rightarrow \infty$ is the scaling parameter. As discussed in Ref. [20], the velocity anisotropy does not enter the RG equations, which remains true at least up to two loops in our fermionic replica trick approach. It is convenient to define $g_{\pm} = g_{3b} \pm g_b$, so that

$$\begin{aligned}\frac{dg}{d \ln s} &= g_3^2 + \frac{1}{2} (g_+^2 + g_-^2), \\ \frac{dg_3}{d \ln s} &= 4 g g_3 + (g_+^2 - g_-^2), \\ \frac{dg_+}{d \ln s} &= 2 (g + g_3) g_+, \\ \frac{dg_-}{d \ln s} &= 2 (g - g_3) g_-.\end{aligned}$$

Given the appropriate bare values of the amplitudes, one readily recognizes that the RG flow maintains the initial condition $g_- = 0$, hence the scaling equations reduce to

$$\frac{dg}{d \ln s} = g_3^2 + \frac{1}{2} g_+^2, \tag{101}$$

$$\frac{dg_3}{d \ln s} = 4 g g_3 + g_+^2, \tag{102}$$

$$\frac{dg_+}{d \ln s} = 2 (g + g_3) g_+. \tag{103}$$

The RG equations with the appropriate initial conditions always flow to strong coupling with

$$g_3 \sim 2 g \sim g_+ \rightarrow \frac{1}{3} \frac{1}{\ln s_c - \ln s},$$

where s_c can be interpreted as the correlation length of the modes which acquire a mass gap E_c by the interaction, with

$$E_c \simeq \frac{\sqrt{v_F v_{\Delta}}}{s_c}. \tag{104}$$

Therefore the two pairs of nodes get strongly coupled, in agreement with Refs. [19,20,22].

B. Strong coupling analysis

In order to gain further insight into the strong coupling phase towards which RG flows, let us consider the case of a single replica $N = 1$. For further simplification it is convenient to adopt the same approach as in Refs. 5,19,20,22 and neglect the role of the opposite frequencies, which amounts to drop the τ -label. The model thus reduces to two interacting chains of spinful fermions, each chain representing a pair of nodes. The coupling g of Fig. 1 only couples to the charge sector and makes the intra-chain umklapp, the coupling g_3 , a relevant perturbation which opens a charge gap on each chain. Therefore the model is equivalent to two coupled spin-1/2 chains. If we denote by $\vec{n}_{1(2)}(x)$ and $\epsilon_{1(2)}(x)$ respectively the staggered magnetization and the dimerization of chain 1(2), the coupling among the chains is ferromagnetic and given by

$$w^2 \int dx \epsilon_1(x) \epsilon_2(x) - 4 \vec{n}_1(x) \cdot \vec{n}_2(x).$$

As shown in Ref. 29, this model is equivalent to an SO(4) Gross-Neveu which turns out to be fully massive or, equivalently, by four two-dimensional off-critical classical Ising models, three ordered and one disordered, $\langle \sigma_1 \rangle = \langle \sigma_2 \rangle = \langle \sigma_3 \rangle = \langle \mu_4 \rangle = \sigma \neq 0$, where σ_i and μ_i , $i = 1, \dots, 4$, are order and disorder parameters, respectively. The ground state is rigid to an external magnetic field and to a spin vector potential opposite for the two chains, hence the conductivity is zero. As we discussed, a finite frequency amounts to add a term

$$\omega \int dx \epsilon_1(x) + \epsilon_2(x) \propto \omega \int dx \sigma_1(x) \sigma_2(x) \sigma_3(x) \sigma_4(x) \simeq \omega \sigma^3 \int dx \sigma_4(x),$$

which actually plays the role of an external magnetic field acting on the fourth disordered Ising copy. The net result is that $\langle \sigma_4(x) \rangle_{\omega \neq 0} \sim \omega$, which in turns mean that the DOS remains linear in frequency. Even in the presence of an explicit dimerization, the susceptibilities towards a magnetic field or towards a spin vector potential opposite for the two chains still vanish. Were these results valid for any N , even for $N \rightarrow 0$, we should conclude that *i*) the model is indeed insulating; *ii*) the DOS is linear in frequency, in full agreement with Ref. 6.

C. Identification of relevant energy scales

The previous analysis of the $N = 1$ model shows that the vanishing of thermal and spin conductivities in the model for a disordered d -wave superconductor translates in the language

of the effective one-dimensional fermionic model into the existence of a finite spin-gap. The correlation length associated with this gap should then represent the localization length of the Bogoliubov quasiparticles. We may estimate this correlation length as the scale s_c at which the RG equations (101-103) encounter a singularity. In addition we may introduce the mass gap E_c through Eq. (104) which can be identified as the energy scale around which localization effects appear. It is worth noticing that s_c underestimates the spin correlation length, hence the actual localization length. The reason is that the RG equations blow up on a scale which is related to the largest gap in the excitation spectrum. Since the coupling g only affects charge degrees of freedom, the largest gap is expected in the charge sector, the spin gap being smaller. Keeping this in mind, in what follows we shall discuss how s_c , or better E_c , depend on the range of the disorder potential.

First of all we need to identify some reference scale to compare with E_c . The natural candidate would be the inverse relaxation time $1/\tau_0$ in the Born approximation. In the generic case $u^2 \geq w^2 \geq v^2$, $1/\tau_0 = 2Q_0$ where Q_0 is obtained by Eq. (56) with u^2 substituted by $u^2 + 2w^2 + v^2$. We expect that the actual E_c is always smaller than $1/\tau_0$, the two values being closest for extremely short-range disorder, namely $u^2 = w^2 = v^2$. Since the derivation of the non-linear σ -model does not provide with the precise dependence of the initial values of the coupling constants, $g^{(0)}$, $g_3^{(0)}$ and $g_+^{(0)}$, on the impurity potential, we will assume that the short range disorder corresponds to $2g^{(0)} = g_3^{(0)} = g_+^{(0)}$ and moreover that the mass gap E_c in this case can be identified as $1/\tau_0$ in the Born approximation. Upon integrating the RG equations (101-103) with this initial condition, the value of $E_c \simeq 1/\tau_0$ is found to be

$$\frac{1}{\tau_0} \simeq \Lambda \exp \left(-\frac{1}{2g^{(0)} + g_3^{(0)} + g_+^{(0)}} \right), \quad (105)$$

where Λ is the ultraviolet cut-off of the order of the gap Δ . In order to appreciate the role of the range of the disorder-potential, let us analyze the RG equations (101-103) keeping fixed the combination $g_0 = 2g^{(0)} + g_3^{(0)} + g_+^{(0)}$, i.e. at constant $1/\tau_0$, and increasing the value of $g^{(0)}$ at expenses of $g_3^{(0)} + g_+^{(0)} = g_0 - 2g^{(0)}$. One readily realizes that as $g^{(0)}$ increases E_c decreases from its short-range value (105). For instance, if we assume $g_3^{(0)} = 0$ and $g_+^{(0)} = g_0 - 2g^{(0)} \ll g_0$, then

$$E_c \simeq \frac{1}{\tau_0} \left(\frac{eg_+^{(0)}}{4g_0} \right)^{\frac{1}{g_0}} \ll \frac{1}{\tau_0},$$

which explicitly shows that localization effects may show up at energies/temperatures much

smaller than the $1/\tau_0$ in the Born approximation.

XI. CONCLUSIONS

In this work we have presented a derivation of the non-linear σ -model for disordered d -wave superconductors able to deal with a finite range impurity potential, namely with an intra-node scattering potential generically different from the inter-nodal one. Within this derivation we have been able to clarify some controversial issues concerning the validity of a conventional non-linear σ -model approach when dealing with disordered Dirac fermions. We have indeed found that the non-linear σ -model approach is actually equivalent to the alternative method first introduced in Ref. 19 which consists in mapping the disordered model onto a one-dimensional model of interacting fermions. The energy upper cut-off is provided by the inverse Born relaxation time $1/\tau_0$ in the non-linear σ -model approach and by the superconducting gap Δ in the 1d mapping. A closely related aspect which also emerges for d -wave superconductors is the existence of a Wess-Zumino-Novikov-Witten term related to the vorticity of the spectrum in momentum space. Both these features are responsible of several interesting phenomena. For instance, unlike conventional disordered systems, in this case disorder starts playing a role (particularly in the density of states) when quasiparticle motion is still ballistic. In contrast, the on-set of localization precursor effects is pushed towards energies/temperatures lower than the inverse relaxation time in the Drude approximation. More specifically, the longer is the range of the impurity potential the later localization effects appear. This result may explain why experiments have so often failed to detect localization precursor effects in cuprates superconductors.

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